Chapter 3

The Real Numbers

3.1 Natural Numbers and Induction.

In this course, we let

$$\mathbb{N} := \{1, 2, 3, \ldots\}$$

denote the set of all natural numbers.

First of all, let us assume the important property of $\mathbb{N}$ as the following axiom.

Axiom 3.1.1: (Well-Ordering Property of $\mathbb{N}$)
[หลักการอันดับอย่างดีของจำนวนนับ]

If S is a nonempty subset of $\mathbb{N}$, then there exists an element $m \in S$ such that $m \le s$ for all $s \in S$.

[If $\emptyset \neq S \subseteq \mathbb{N}$, then $\exists m \in S \ni m \le s, \forall s \in S$]

Example: ① $S = \{2, 4, 6, 8, \ldots, 2n, \ldots\}$

Consider $S \neq \emptyset$, $S \subseteq \mathbb{N}$,
we have $\exists m = 2 \in S \ni \forall s \in S, m \leq s.$

② $S = \{10{,}000{,}000, 10{,}000{,}001, \ldots\}$

เครื่องมืออย่างหนึ่งที่มีความสำคัญอย่างมากในการพิสูจน์ทฤษฎีบททางคณิตศาสตร์ ซึ่งเกี่ยวข้องกับจำนวนนับ คือ หลักการอุปนัยเชิงคณิตศาสตร์ มีรายละเอียดดังนี้

Theorem 3.1.2 [Principle of Mathematical Induction]
[หลักการอุปนัยเชิงคณิตศาสตร์]

Let $P(n)$ be a statement for all $n \in \mathbb{N}$.
Suppose that:

(1) $P(1)$ is true.

(2) For each $k \in \mathbb{N}$, if $P(k)$ is true, then $P(k+1)$ is also true.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

<u>Proof</u>. $[(p \wedge q) \Rightarrow r \Leftrightarrow (p \wedge q) \wedge \sim r \Rightarrow \lightning]$

Suppose that the hypotheses (1) and (2) hold true, but $P(n)$ is <u>not true</u> for some $n \in \mathbb{N}$.
false

Now, we let
$$S := \{n \in \mathbb{N} : P(n) \text{ is false}\}.$$
Since we suppose that $P(n)$ is false for some $n \in \mathbb{N}$, we can ensure that $S \neq \emptyset$.

Then, the well-ordering principle of $\mathbb{N}$ guarantees that there exists an element $m \in S$ such that $m \leq s$ for all $s \in S$.
Since $m \in S$, we note that $P(m)$ is false.

On the other hand, we know that $P(1)$ is true, it follows that $1 \notin S$. This means that $m > 1$. (Why?)

Since $m \in \mathbb{N}$ and $m > 1$, we know that $m-1 \in \mathbb{N}$ and $m-1 \notin S$. This implies that $P(m-1)$ is true. By using (2), we obtain that $P(m-1+1)$ is also true, that is $P(m)$ is true. Thus, we get that $m \notin S$, which leads to a contradiction with $m \in S$.

Therefore, we conclude that $P(n)$ must be true for all $n \in \mathbb{N}$.

$\square$

ในการพิสูจน์ใช้ Principle of Mathematical Induction
เราต้องตรวจสอบค่าความจริงสองประการ

1. Basis for Induction
2. Induction Step

Example 3.1.3: Prove that

$$1+2+3+\cdots+n = \frac{1}{2}n(n+1) \text{ for every } n \in \mathbb{N}.$$

Proof. Let $P(n)$ be the statement
"$1+2+3+\dots+n = \frac{1}{2}n(n+1)$"

Basis for induction: Let us consider "$P(1)$".
Note that

$$1 = \frac{1}{2}(1)(1+1),$$

this means that $P(1)$ is true.

Induction Step: Let k be given and assume that $P(k)$ holds. That is,

$$1+2+\dots+k = \frac{1}{2}k(k+1).$$

We claim that $P(k+1)$ is also true.
Consider,

$$\underbrace{1+2+\dots+k}_{=\frac{1}{2}k(k+1)} + (k+1) = \frac{1}{2}k(k+1)+(k+1)$$

$$= \frac{1}{2}\left[k(k+1)+2(k+1)\right]$$

$$= \frac{1}{2}(k+1)(k+2)$$

$$= \frac{1}{2}(k+1)((k+1)+1),$$

which means that $P(k+1)$ is also true.
Therefore, by using the principle of mathematical induction, we conclude that $P(n)$ is true for all $n \in \mathbb{N}$.

$\square$

ทฤษฎีบทต่อไปนี้จะเป็นการวางนัยทั่วไปของ PMI
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Theorem 3.1.6:

Let $m \in \mathbb{N}$ and let $P(n)$ be a statement for each $n \geq m$. Suppose that:

(1) $P(m)$ is true.

(2) For each $k \geq m$, if $P(k)$ is true, then also $P(k+1)$ is also true.

Then $P(n)$ is true for all $n \geq m$.

Proof. [Exercise!]

Example: Prove that $2^n > 2n+1$ for all $n \geq 3$.

Proof.

For $n = 3$, we note that

$$2^3 = 8 > 7 = 2(3)+1,$$

which implies that the statement holds for $n=3$.

Let $k \geq 3$ be given and suppose that

$$2^k > 2k+1.$$

We claim that $2^{k+1} > 2(k+1)+1$. [$2k+2+1$]

Consider,

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &> 2(2k+1) \\ &= 4k+2 \\ &= 2k+2+2k > 2k+2+1 \end{aligned}$$

$$= 2(k+1)+1,$$

which means that $P(k+1)$ is also true.

Hence, we conclude that $P(n)$ is true for all $n \geq 3$.

$\square$