QUICK CHECK EXERCISES 7.1 (See page 491 for answers.)

1. Use algebraic manipulation and (if necessary) u-substitution to integrate the function.
   (a) $\int \frac{x+1}{x}\,dx =$ ______
   (b) $\int \frac{x+2}{x+1}\,dx =$ ______
   (c) $\int \frac{2x+1}{x^2+1}\,dx =$ ______
   (d) $\int xe^{3\ln x}\,dx =$ ______

2. Use trigonometric identities and (if necessary) u-substitution to integrate the function.
   (a) $\int \frac{1}{\csc x}\,dx =$ ______
   (b) $\int \frac{1}{\cos^2 x}\,dx =$ ______
   (c) $\int (\cot^2 x + 1)\,dx =$ ______
   (d) $\int \frac{1}{\sec x + \tan x}\,dx =$ ______

3. Integrate the function.
   (a) $\int \sqrt{x-1}\,dx =$ ______
   (b) $\int e^{2x+1}\,dx =$ ______
   (c) $\int (\sin^3 x \cos x + \sin x \cos^3 x)\,dx =$ ______
   (d) $\int \frac{1}{(e^x + e^{-x})^2}\,dx =$ ______

EXERCISE SET 7.1

1–30 Evaluate the integrals by making appropriate u-substitutions and applying the formulas reviewed in this section.

1. $\int (4-2x)^3\,dx$
2. $\int 3\sqrt{4+2x}\,dx$
3. $\int x\sec^2(x^2)\,dx$
4. $\int 4x\tan(x^2)\,dx$
5. $\int \frac{\sin 3x}{2+\cos 3x}\,dx$
6. $\int \frac{1}{9+4x^2}\,dx$
7. $\int e^x \sinh(e^x)\,dx$
8. $\int \frac{\sec(\ln x)\tan(\ln x)}{x}\,dx$
9. $\int e^{\tan x}\sec^2 x\,dx$
10. $\int \frac{x}{\sqrt{1-x^4}}\,dx$
11. $\int \cos^5 5x \sin 5x\,dx$
12. $\int \frac{\cos x}{\sin x\sqrt{\sin^2 x + 1}}\,dx$
13. $\int \frac{e^x}{\sqrt{4+e^{2x}}}\,dx$
14. $\int \frac{e^{\tan^{-1} x}}{1+x^2}\,dx$
15. $\int \frac{e^{\sqrt{x-1}}}{\sqrt{x-1}}\,dx$
16. $\int (x+1)\cot(x^2+2x)\,dx$
17. $\int \frac{\cosh\sqrt{x}}{\sqrt{x}}\,dx$
18. $\int \frac{dx}{x(\ln x)^2}$
19. $\int \frac{dx}{\sqrt{x}\,3^{\sqrt{x}}}$

20. $\int \sec(\sin\theta)\tan(\sin\theta)\cos\theta\,d\theta$
21. $\int \frac{\operatorname{csch}^2(2/x)}{x^2}\,dx$
22. $\int \frac{dx}{\sqrt{x^2-4}}$
23. $\int \frac{e^{-x}}{4-e^{-2x}}\,dx$
24. $\int \frac{\cos(\ln x)}{x}\,dx$
25. $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$
26. $\int \frac{\sinh(x^{-1/2})}{x^{3/2}}\,dx$
27. $\int \frac{x}{\csc(x^2)}\,dx$
28. $\int \frac{e^x}{\sqrt{4-e^{2x}}}\,dx$
29. $\int x4^{-x^2}\,dx$
30. $\int 2^{\pi x}\,dx$

FOCUS ON CONCEPTS

31. (a) Evaluate the integral $\int \sin x \cos x\,dx$ using the substitution $u = \sin x$.
    (b) Evaluate the integral $\int \sin x \cos x\,dx$ using the identity $\sin 2x = 2\sin x \cos x$.
    (c) Explain why your answers to parts (a) and (b) are consistent.

32. (a) Derive the identity
$$\frac{\operatorname{sech}^2 x}{1+\tanh^2 x} = \operatorname{sech} 2x$$
    (b) Use the result in part (a) to evaluate $\int \operatorname{sech} x\,dx$.
    (c) Derive the identity
$$\operatorname{sech} x = \frac{2e^x}{e^{2x}+1}$$
    (d) Use the result in part (c) to evaluate $\int \operatorname{sech} x\,dx$.
    (e) Explain why your answers to parts (b) and (d) are consistent.

33. (a) Derive the identity
$$\frac{\sec^2 x}{\tan x} = \frac{1}{\sin x \cos x}$$
    (b) Use the identity $\sin 2x = 2\sin x \cos x$ along with the result in part (a) to evaluate $\int \csc x\,dx$.
    (c) Use the identity $\cos x = \sin[(\pi/2) - x]$ along with your answer to part (a) to evaluate $\int \sec x\,dx$.

QUICK CHECK EXERCISES 7.2 *(See page 500 for answers.)*

1. (a) If $G'(x) = g(x)$, then
$$\int f(x)g(x)\,dx = f(x)G(x) - \underline{\qquad}$$
(b) If $u = f(x)$ and $v = G(x)$, then the formula in part (a) can be written in the form $\int u\,dv = \underline{\qquad}$.

2. Find an appropriate choice of u and dv for integration by parts of each integral. Do not evaluate the integral.
(a) $\int x \ln x\,dx$; $u = \underline{\qquad}$, $dv = \underline{\qquad}$
(b) $\int (x-2)\sin x\,dx$; $u = \underline{\qquad}$, $dv = \underline{\qquad}$
(c) $\int \sin^{-1} x\,dx$; $u = \underline{\qquad}$, $dv = \underline{\qquad}$
(d) $\int \frac{x}{\sqrt{x-1}}\,dx$; $u = \underline{\qquad}$, $dv = \underline{\qquad}$

3. Use integration by parts to evaluate the integral.
(a) $\int xe^{2x}\,dx$ (b) $\int \ln(x-1)\,dx$
(c) $\int_0^{\pi/6} x \sin 3x\,dx$

4. Use a reduction formula to evaluate $\int \sin^3 x\,dx$.

EXERCISE SET 7.2

1–38 Evaluate the integral.

1. $\int xe^{-2x}\,dx$
2. $\int xe^{3x}\,dx$
3. $\int x^2e^x\,dx$
4. $\int x^2e^{-2x}\,dx$
5. $\int x \sin 3x\,dx$
6. $\int x \cos 2x\,dx$
7. $\int x^2 \cos x\,dx$
8. $\int x^2 \sin x\,dx$
9. $\int x \ln x\,dx$
10. $\int \sqrt{x} \ln x\,dx$
11. $\int (\ln x)^2\,dx$
12. $\int \frac{\ln x}{\sqrt{x}}\,dx$
13. $\int \ln(3x-2)\,dx$
14. $\int \ln(x^2+4)\,dx$
15. $\int \sin^{-1} x\,dx$
16. $\int \cos^{-1}(2x)\,dx$
17. $\int \tan^{-1}(3x)\,dx$
18. $\int x \tan^{-1} x\,dx$
19. $\int e^x \sin x\,dx$
20. $\int e^{3x} \cos 2x\,dx$
21. $\int \sin(\ln x)\,dx$
22. $\int \cos(\ln x)\,dx$
23. $\int x \sec^2 x\,dx$
24. $\int x \tan^2 x\,dx$
25. $\int x^3e^{x^2}\,dx$
26. $\int \frac{xe^x}{(x+1)^2}\,dx$
27. $\int_0^2 xe^{2x}\,dx$
28. $\int_0^1 xe^{-5x}\,dx$
29. $\int_1^e x^2 \ln x\,dx$
30. $\int_{\sqrt{e}}^e \frac{\ln x}{x^2}\,dx$
31. $\int_{-1}^1 \ln(x+2)\,dx$
32. $\int_0^{\sqrt{3}/2} \sin^{-1} x\,dx$
33. $\int_2^4 \sec^{-1}\sqrt{\theta}\,d\theta$
34. $\int_1^2 x \sec^{-1} x\,dx$
35. $\int_0^{\pi} x \sin 2x\,dx$
36. $\int_0^{\pi} (x + x\cos x)\,dx$
37. $\int_1^3 \sqrt{x}\tan^{-1}\sqrt{x}\,dx$
38. $\int_0^2 \ln(x^2+1)\,dx$

39–42 True–False Determine whether the statement is true or false. Explain your answer.

39. The main goal in integration by parts is to choose u and dv to obtain a new integral that is easier to evaluate than the original.

40. Applying the LIATE strategy to evaluate $\int x^3 \ln x\,dx$, we should choose $u = x^3$ and $dv = \ln x\,dx$.

41. To evaluate $\int \ln e^x\,dx$ using integration by parts, choose $dv = e^x\,dx$.

42. Tabular integration by parts is useful for integrals of the form $\int p(x)f(x)\,dx$, where $p(x)$ is a polynomial and $f(x)$ can be repeatedly integrated.

43–44 Evaluate the integral by making a u-substitution and then integrating by parts.

43. $\int e^{\sqrt{x}}\,dx$
44. $\int \cos\sqrt{x}\,dx$

45. Prove that tabular integration by parts gives the correct answer for
$$\int p(x)f(x)\,dx$$
where $p(x)$ is any *quadratic polynomial* and $f(x)$ is any function that can be repeatedly integrated.

46. The computations of any integral evaluated by repeated integration by parts can be organized using tabular integration by parts. Use this organization to evaluate $\int e^x \cos x\,dx$ in

two ways: first by repeated differentiation of $\cos x$ (compare Example 5), and then by repeated differentiation of e^x.

47–52 Evaluate the integral using [redacted] integration by parts. ■

47. $\int (3x^2 - x + 2)e^{-x}\,dx$

48. $\int (x^2 + x + 1)\sin x\,dx$

49. $\int 4x^4 \sin 2x\,dx$

50. $\int x^3\sqrt{2x+1}\,dx$

51. $\int e^{ax}\sin bx\,dx$

52. $\int e^{-3\theta}\sin 5\theta\,d\theta$

53. Consider the integral $\int \sin x \cos x\,dx$.
(a) Evaluate the integral two ways: first using integration by parts, and then using the substitution $u = \sin x$.
(b) Show that the results of part (a) are equivalent.
(c) Which of the two methods do you prefer? Discuss the reasons for your preference.

54. Evaluate the integral

$$\int_0^1 \frac{x^3}{\sqrt{x^2+1}}\,dx$$

using
(a) integration by parts
(b) the substitution $u = \sqrt{x^2+1}$.

55. (a) Find the area of the region enclosed by $y = \ln x$, the line $x = e$, and the x-axis.
(b) Find the volume of the solid generated when the region in part (a) is revolved about the x-axis.

56. Find the area of the region between $y = x\sin x$ and $y = x$ for $0 \le x \le \pi/2$.

57. Find the volume of the solid generated when the region between $y = \sin x$ and $y = 0$ for $0 \le x \le \pi$ is revolved about the y-axis.

58. Find the volume of the solid generated when the region enclosed between $y = \cos x$ and $y = 0$ for $0 \le x \le \pi/2$ is revolved about the y-axis.

59. A particle moving along the x-axis has velocity function $v(t) = t^3\sin t$. How far does the particle travel from time $t = 0$ to $t = \pi$?

60. The study of sawtooth waves in electrical engineering leads to integrals of the form

$$\int_{-\pi/\omega}^{\pi/\omega} t\sin(k\omega t)\,dt$$

where k is an integer and ω is a nonzero constant. Evaluate the integral.

61. Use reduction formula [redacted] to evaluate
(a) $\int \sin^4 x\,dx$ (b) $\int_0^{\pi/2} \sin^5 x\,dx$.

62. Use reduction formula [redacted]) to evaluate
(a) $\int \cos^5 x\,dx$ (b) $\int_0^{\pi/2} \cos^6 x\,dx$.

63. Derive reduction formula (9).

64. In each part, use integration by parts or other methods to derive the reduction formula.

(a) $\displaystyle\int \sec^n x\,dx = \frac{\sec^{n-2}x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx$

(b) $\displaystyle\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx$

(c) $\displaystyle\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$

65–66 Use the reduction formulas in Exercise 64 to evaluate the integrals. ■

65. (a) $\int \tan^4 x\,dx$ (b) $\int \sec^4 x\,dx$ (c) $\int x^3 e^x\,dx$

66. (a) $\int x^2 e^{3x}\,dx$ (b) $\int_0^1 xe^{-\sqrt{x}}\,dx$
[*Hint:* First make a substitution.]

67. Let f be a function whose second derivative is continuous on $[-1, 1]$. Show that

$$\int_{-1}^{1} xf''(x)\,dx = f'(1) + f'(-1) - f(1) + f(-1)$$

FOCUS ON CONCEPTS

68. (a) In the integral $\int x\cos x\,dx$, let

$$u = x, \quad dv = \cos x\,dx,$$
$$du = dx, \quad v = \sin x + C_1$$

Show that the constant C_1 cancels out, thus giving the same solution obtained by omitting C_1.
(b) Show that in general

$$uv - \int v\,du = u(v + C_1) - \int (v + C_1)\,du$$

thereby justifying the omission of the constant of integration when calculating v in integration by parts.

69. Evaluate $\int \ln(x+1)\,dx$ using integration by parts. Simplify the computation of $\int v\,du$ by introducing a constant of integration $C_1 = 1$ when going from dv to v.

70. Evaluate $\int \ln(3x-2)\,dx$ using integration by parts. Simplify the computation of $\int v\,du$ by introducing a constant of integration $C_1 = -\frac{2}{3}$ when going from dv to v. Compare your solution with your answer to Exercise 13.

71. Evaluate $\int x\tan^{-1}x\,dx$ using integration by parts. Simplify the computation of $\int v\,du$ by introducing a constant of integration $C_1 = \frac{1}{2}$ when going from dv to v.

72. What equation results if integration by parts is applied to the integral

$$\int \frac{1}{x\ln x}\,dx$$

with the choices

$$u = \frac{1}{\ln x} \quad \text{and} \quad dv = \frac{1}{x}\,dx?$$

In what sense is this equation true? In what sense is it false?

✔ QUICK CHECK EXERCISES 7.3 (See page 508 for answers.)

1. Complete each trigonometric identity with an expression involving $\cos 2x$.
(a) $\sin^2 x =$ ______ (b) $\cos^2 x =$ ______
(c) $\cos^2 x - \sin^2 x =$ ______

2. Evaluate the integral.
(a) $\int \sec^2 x\,dx =$ ______
(b) $\int \tan^2 x\,dx =$ ______
(c) $\int \sec x\,dx =$ ______
(d) $\int \tan x\,dx =$ ______

3. Use the indicated substitution to rewrite the integral in terms of u. Do not evaluate the integral.
(a) $\int \sin^2 x \cos x\,dx;\ u = \sin x$
(b) $\int \sin^3 x \cos^2 x\,dx;\ u = \cos x$
(c) $\int \tan^3 x \sec^2 x\,dx;\ u = \tan x$
(d) $\int \tan^3 x \sec x\,dx;\ u = \sec x$

EXERCISE SET 7.3

1–52 Evaluate the integral. ■

1. $\int \cos^3 x \sin x\,dx$
2. $\int \sin^5 3x \cos 3x\,dx$
3. $\int \sin^2 5\theta\,d\theta$
4. $\int \cos^2 3x\,dx$
5. $\int \sin^3 a\theta\,d\theta$
6. $\int \cos^3 at\,dt$
7. $\int \sin ax \cos ax\,dx$
8. $\int \sin^3 x \cos^3 x\,dx$
9. $\int \sin^2 t \cos^3 t\,dt$
10. $\int \sin^3 x \cos^2 x\,dx$
11. $\int \sin^2 x \cos^2 x\,dx$
12. $\int \sin^2 x \cos^4 x\,dx$
13. $\int \sin 2x \cos 3x\,dx$
14. $\int \sin 3\theta \cos 2\theta\,d\theta$
15. $\int \sin x \cos(x/2)\,dx$
16. $\int \cos^{1/3} x \sin x\,dx$
17. $\int_0^{\pi/2} \cos^3 x\,dx$
18. $\int_0^{\pi/2} \sin^2 \frac{x}{2} \cos^2 \frac{x}{2}\,dx$
19. $\int_0^{\pi/3} \sin^4 3x \cos^3 3x\,dx$
20. $\int_{-\pi}^{\pi} \cos^2 5\theta\,d\theta$
21. $\int_0^{\pi/6} \sin 4x \cos 2x\,dx$
22. $\int_0^{2\pi} \sin^2 kx\,dx$
23. $\int \sec^2(2x-1)\,dx$
24. $\int \tan 5x\,dx$
25. $\int e^{-x}\tan(e^{-x})\,dx$
26. $\int \cot 3x\,dx$
27. $\int \sec 4x\,dx$
28. $\int \frac{\sec(\sqrt{x})}{\sqrt{x}}\,dx$
29. $\int \tan^2 x \sec^2 x\,dx$
30. $\int \tan^5 x \sec^4 x\,dx$
31. $\int \tan 4x \sec^4 4x\,dx$
32. $\int \tan^4 \theta \sec^4 \theta\,d\theta$
33. $\int \sec^5 x \tan^3 x\,dx$
34. $\int \tan^5 \theta \sec \theta\,d\theta$
35. $\int \tan^4 x \sec x\,dx$
36. $\int \tan^2 x \sec^3 x\,dx$
37. $\int \tan t \sec^3 t\,dt$
38. $\int \tan x \sec^5 x\,dx$
39. $\int \sec^4 x\,dx$
40. $\int \sec^5 x\,dx$
41. $\int \tan^3 4x\,dx$
42. $\int \tan^4 x\,dx$
43. $\int \sqrt{\tan x}\sec^4 x\,dx$
44. $\int \tan x \sec^{3/2} x\,dx$
45. $\int_0^{\pi/8} \tan^2 2x\,dx$
46. $\int_0^{\pi/6} \sec^3 2\theta \tan 2\theta\,d\theta$
47. $\int_0^{\pi/2} \tan^5 \frac{x}{2}\,dx$
48. $\int_0^{1/4} \sec \pi x \tan \pi x\,dx$
49. $\int \cot^3 x \csc^3 x\,dx$
50. $\int \cot^2 3t \sec 3t\,dt$
51. $\int \cot^3 x\,dx$
52. $\int \csc^4 x\,dx$

53–56 True–False Determine whether the statement is true or false. Explain your answer. ■

53. To evaluate $\int \sin^5 x \cos^8 x\,dx$, use the trigonometric identity $\sin^2 x = 1 - \cos^2 x$ and the substitution $u = \cos x$.

54. To evaluate $\int \sin^8 x \cos^5 x\,dx$, use the trigonometric identity $\sin^2 x = 1 - \cos^2 x$ and the substitution $u = \cos x$.

55. The trigonometric identity
$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha-\beta) + \sin(\alpha+\beta)]$$
is often useful for evaluating integrals of the form $\int \sin^m x \cos^n x\,dx$.

56. The integral $\int \tan^4 x \sec^5 x\,dx$ is equivalent to one whose integrand is a polynomial in $\sec x$.

57. Let m, n be distinct nonnegative integers. Use Formulas (16)–(18) to prove:
(a) $\int_0^{2\pi} \sin mx \cos nx\,dx = 0$
(b) $\int_0^{2\pi} \cos mx \cos nx\,dx = 0$
(c) $\int_0^{2\pi} \sin mx \sin nx\,dx = 0.$

58. Evaluate the integrals in Exercise 57 when m and n denote the *same* nonnegative integer.

59. Find the arc length of the curve $y = \ln(\cos x)$ over the interval $[0, \pi/4]$.

60. Find the volume of the solid generated when the region enclosed by $y = \tan x$, $y = 1$, and $x = 0$ is revolved about the x-axis.

61. Find the volume of the solid that results when the region enclosed by $y = \cos x$, $y = \sin x$, $x = 0$, and $x = \pi/4$ is revolved about the x-axis.

62. The region bounded below by the x-axis and above by the portion of $y = \sin x$ from $x = 0$ to $x = \pi$ is revolved about the x-axis. Find the volume of the resulting solid.

63. Use Formula (27) to show that if the length of the equatorial line on a Mercator projection is L, then the vertical distance D between the latitude lines at $\alpha°$ and $\beta°$ on the same side of the equator (where $\alpha < \beta$) is
$$D = \frac{L}{2\pi}\ln\left|\frac{\sec\beta° + \tan\beta°}{\sec\alpha° + \tan\alpha°}\right|$$

64. Suppose that the equator has a length of 100 cm on a Mercator projection. In each part, use the result in Exercise 63 to answer the question.
(a) What is the vertical distance on the map between the equator and the line at 25° north latitude?
(b) What is the vertical distance on the map between New Orleans, Louisiana, at 30° north latitude and Winnipeg, Canada, at 50° north latitude?

FOCUS ON CONCEPTS

65. (a) Show that
$$\int \csc x\,dx = -\ln|\csc x + \cot x| + C$$
(b) Show that the result in part (a) can also be written as
$$\int \csc x\,dx = \ln|\csc x - \cot x| + C$$
and
$$\int \csc x\,dx = \ln\left|\tan \tfrac{1}{2}x\right| + C$$

QUICK CHECK EXERCISES 7.4

(See page 514 for answers.)

1. For each expression, give a trigonometric substitution that will eliminate the radical.
(a) $\sqrt{a^2 - x^2}$ ________ (b) $\sqrt{a^2 + x^2}$ ________
(c) $\sqrt{x^2 - a^2}$ ________

2. If $x = 2\sec\theta$ and $0 < \theta < \pi/2$, then
(a) $\sin\theta =$ ________ (b) $\cos\theta =$ ________
(c) $\tan\theta =$ ________.

3. In each part, state the trigonometric substitution that you would try first to evaluate the integral. Do not evaluate the integral.
(a) $\int \sqrt{9 + x^2}\, dx$ ________
(b) $\int \sqrt{9 - x^2}\, dx$ ________
(c) $\int \sqrt{1 - 9x^2}\, dx$ ________
(d) $\int \sqrt{x^2 - 9}\, dx$ ________
(e) $\int \sqrt{9 + 3x^2}\, dx$ ________
(f) $\int \sqrt{1 + (9x)^2}\, dx$ ________

4. In each part, determine the substitution u.
(a) $\int \dfrac{1}{x^2 - 2x + 10}\, dx = \int \dfrac{1}{u^2 + 3^2}\, du$;
$u =$ ________
(b) $\int \sqrt{x^2 - 6x + 8}\, dx = \int \sqrt{u^2 - 1}\, du$;
$u =$ ________
(c) $\int \sqrt{12 - 4x - x^2}\, dx = \int \sqrt{4^2 - u^2}\, du$;
$u =$ ________

EXERCISE SET 7.4 C CAS

1–26 Evaluate the integral.

1. $\int \sqrt{4 - x^2}\, dx$
2. $\int \sqrt{1 - 4x^2}\, dx$
3. $\int \dfrac{x^2}{\sqrt{16 - x^2}}\, dx$
4. $\int \dfrac{dx}{x^2\sqrt{9 - x^2}}$
5. $\int \dfrac{dx}{(4 + x^2)^2}$
6. $\int \dfrac{x^2}{\sqrt{5 + x^2}}\, dx$
7. $\int \dfrac{\sqrt{x^2 - 9}}{x}\, dx$
8. $\int \dfrac{dx}{x^2\sqrt{x^2 - 16}}$
9. $\int \dfrac{3x^3}{\sqrt{1 - x^2}}\, dx$
10. $\int x^3\sqrt{5 - x^2}\, dx$
11. $\int \dfrac{dx}{x^2\sqrt{9x^2 - 4}}$
12. $\int \dfrac{\sqrt{1 + t^2}}{t}\, dt$
13. $\int \dfrac{dx}{(1 - x^2)^{3/2}}$
14. $\int \dfrac{dx}{x^2\sqrt{x^2 + 25}}$
15. $\int \dfrac{dx}{\sqrt{x^2 - 9}}$
16. $\int \dfrac{dx}{1 + 2x^2 + x^4}$
17. $\int \dfrac{dx}{(4x^2 - 9)^{3/2}}$
18. $\int \dfrac{3x^3}{\sqrt{x^2 - 25}}\, dx$
19. $\int e^x\sqrt{1 - e^{2x}}\, dx$
20. $\int \dfrac{\cos\theta}{\sqrt{2 - \sin^2\theta}}\, d\theta$
21. $\int_0^1 5x^3\sqrt{1 - x^2}\, dx$
22. $\int_0^{1/2} \dfrac{dx}{(1 - x^2)^2}$
23. $\int_{\sqrt{2}}^{2} \dfrac{dx}{x^2\sqrt{x^2 - 1}}$
24. $\int_{\sqrt{2}}^{2} \dfrac{\sqrt{2x^2 - 4}}{x}\, dx$
25. $\int_1^3 \dfrac{dx}{x^4\sqrt{x^2 + 3}}$
26. $\int_0^3 \dfrac{x^3}{(3 + x^2)^{5/2}}\, dx$

27–30 True–False Determine whether the statement is true or false. Explain your answer.

27. An integrand involving a radical of the form $\sqrt{a^2 - x^2}$ suggests the substitution $x = a\sin\theta$.
28. The trigonometric substitution $x = a\sin\theta$ is made with the restriction $0 \le \theta \le \pi$.
29. An integrand involving a radical of the form $\sqrt{x^2 - a^2}$ suggests the substitution $x = a\cos\theta$.
30. The area enclosed by the ellipse $x^2 + 4y^2 = 1$ is $\pi/2$.

FOCUS ON CONCEPTS

31. The integral
$$\int \frac{x}{x^2 + 4}\, dx$$
can be evaluated either by a trigonometric substitution or by the substitution $u = x^2 + 4$. Do it both ways and show that the results are equivalent.

32. The integral
$$\int \frac{x^2}{x^2 + 4}\, dx$$
can be evaluated either by a trigonometric substitution or by algebraically rewriting the numerator of the integrand as $(x^2 + 4) - 4$. Do it both ways and show that the results are equivalent.

33. Find the arc length of the curve $y = \ln x$ from $x = 1$ to $x = 2$.

34. Find the arc length of the curve $y = x^2$ from $x = 0$ to $x = 1$.

35. Find the area of the surface generated when the curve in Exercise 34 is revolved about the x-axis.

36. Find the volume of the solid generated when the region enclosed by $x = y(1 - y^2)^{1/4}$, $y = 0$, $y = 1$, and $x = 0$ is revolved about the y-axis.

37–48 Evaluate the integral. ■

37. $\int \frac{dx}{x^2 - 4x + 5}$

38. $\int \frac{dx}{\sqrt{2x - x^2}}$

39. $\int \frac{dx}{\sqrt{3 + 2x - x^2}}$

40. $\int \frac{dx}{16x^2 + 16x + 5}$

41. $\int \frac{dx}{\sqrt{x^2 - 6x + 10}}$

42. $\int \frac{x}{x^2 + 2x + 2}\,dx$

43. $\int \sqrt{3 - 2x - x^2}\,dx$

44. $\int \frac{e^x}{\sqrt{1 + e^x + e^{2x}}}\,dx$

45. $\int \frac{dx}{2x^2 + 4x + 7}$

46. $\int \frac{2x + 3}{4x^2 + 4x + 5}\,dx$

47. $\int_1^2 \frac{dx}{\sqrt{4x - x^2}}$

48. $\int_0^4 \sqrt{x(4 - x)}\,dx$

C **49–50** There is a good chance that your CAS will not be able to evaluate these integrals as stated. If this is so, make a substitution that converts the integral into one that your CAS can evaluate. ■

49. $\int \cos x \sin x \sqrt{1 - \sin^4 x}\,dx$

50. $\int (x \cos x + \sin x)\sqrt{1 + x^2 \sin^2 x}\,dx$

51. (a) Use the ***hyperbolic substitution*** $x = 3\sinh u$, the identity $\cosh^2 u - \sinh^2 u = 1$, and Theorem 6.9.4 to evaluate

$$\int \frac{dx}{\sqrt{x^2 + 9}}$$

(b) Evaluate the integral in part (a) using a trigonometric substitution and show that the result agrees with that obtained in part (a).

52. Use the hyperbolic substitutionn $x = \cosh u$, the identity $\sinh^2 u = \frac{1}{2}(\cosh 2u - 1)$, and the results referenced in Exercise 51 to evaluate

$$\int \sqrt{x^2 - 1}\,dx, \quad x \geq 1$$

53. **Writing** The trigonometric substitution $x = a\sin\theta$, $-\pi/2 \leq \theta \leq \pi/2$, is suggested for an integral whose integrand involves $\sqrt{a^2 - x^2}$. Discuss the implications of restricting θ to $\pi/2 \leq \theta \leq 3\pi/2$, and explain why the restriction $-\pi/2 \leq \theta \leq \pi/2$ should be preferred.

54. **Writing** The trigonometric substitution $x = a\cos\theta$ could also be used for an integral whose integrand involves $\sqrt{a^2 - x^2}$. Determine an appropriate restriction for θ with the substitution $x = a\cos\theta$, and discuss how to apply this substitution in appropriate integrals. Illustrate your discussion by evaluating the integral in Example 1 using a substitution of this type.

QUICK CHECK EXERCISES 7.5 *(See page 523 for answers.)*

1. A partial fraction is a rational function of the form ________ or of the form ________.

2. (a) What is a proper rational function?
   (b) What condition must the degree of the numerator and the degree of the denominator of a rational function satisfy for the method of partial fractions to be applicable directly?
   (c) If the condition in part (b) is not satisfied, what must you do if you want to use partial fractions?

3. Suppose that the function $f(x) = P(x)/Q(x)$ is a proper rational function.
   (a) For each factor of $Q(x)$ of the form $(ax + b)^m$, the partial fraction decomposition of f contains the following sum of m partial fractions: ________
   (b) For each factor of $Q(x)$ of the form $(ax^2 + bx + c)^m$, where $ax^2 + bx + c$ is an irreducible quadratic, the partial fraction decomposition of f contains the following sum of m partial fractions: ________

4. Complete the partial fraction decomposition.
   (a) $\frac{-3}{(x + 1)(2x - 1)} = \frac{A}{x + 1} - \frac{2}{2x - 1}$
   (b) $\frac{2x^2 - 3x}{(x^2 + 1)(3x + 2)} = \frac{B}{3x + 2} - \frac{1}{x^2 + 1}$

5. Evaluate the integral.
   (a) $\int \frac{3}{(x + 1)(1 - 2x)}\,dx$ (b) $\int \frac{2x^2 - 3x}{(x^2 + 1)(3x + 2)}\,dx$

EXERCISE SET 7.5 C CAS

1–8 Write out the form of the partial fraction decomposition. (Do not find the numerical values of the coefficients.) ■

1. $\frac{3x - 1}{(x - 3)(x + 4)}$

2. $\frac{5}{x(x^2 - 4)}$

3. $\frac{2x - 3}{x^3 - x^2}$

4. $\frac{x^2}{(x + 2)^3}$

5. $\frac{1 - x^2}{x^3(x^2 + 2)}$

6. $\frac{3x}{(x - 1)(x^2 + 6)}$

7. $\frac{4x^3 - x}{(x^2 + 5)^2}$

8. $\frac{1 - 3x^4}{(x - 2)(x^2 + 1)^2}$

9–34 Evaluate the integral. ■

9. $\int \frac{dx}{x^2 - 3x - 4}$

10. $\int \frac{dx}{x^2 - 6x - 7}$

11. $\int \frac{11x + 17}{2x^2 + 7x - 4}\,dx$

12. $\int \frac{5x - 5}{3x^2 - 8x - 3}\,dx$

13. $\int \frac{2x^2 - 9x - 9}{x^3 - 9x}\,dx$

14. $\int \frac{dx}{x(x^2 - 1)}$

15. $\int \frac{x^2 - 8}{x + 3}\,dx$

16. $\int \frac{x^2 + 1}{x - 1}\,dx$

17. $\int \frac{3x^2 - 10}{x^2 - 4x + 4}\,dx$

18. $\int \frac{x^2}{x^2 - 3x + 2}\,dx$

19. $\int \frac{2x - 3}{x^2 - 3x - 10}\,dx$

20. $\int \frac{3x + 1}{3x^2 + 2x - 1}\,dx$

21. $\int \frac{x^5 + x^2 + 2}{x^3 - x}\,dx$

22. $\int \frac{x^5 - 4x^3 + 1}{x^3 - 4x}\,dx$

23. $\int \frac{2x^2 + 3}{x(x - 1)^2}\,dx$

24. $\int \frac{3x^2 - x + 1}{x^3 - x^2}\,dx$

25. $\displaystyle\int \frac{2x^2-10x+4}{(x+1)(x-3)^2}\,dx$

26. $\displaystyle\int \frac{2x^2-2x-1}{x^3-x^2}\,dx$

27. $\displaystyle\int \frac{x^2}{(x+1)^3}\,dx$

28. $\displaystyle\int \frac{2x^2+3x+3}{(x+1)^3}\,dx$

29. $\displaystyle\int \frac{2x^2-1}{(4x-1)(x^2+1)}\,dx$

30. $\displaystyle\int \frac{dx}{x^3+2x}$

31. $\displaystyle\int \frac{x^3+3x^2+x+9}{(x^2+1)(x^2+3)}\,dx$

32. $\displaystyle\int \frac{x^3+x^2+x+2}{(x^2+1)(x^2+2)}\,dx$

33. $\displaystyle\int \frac{x^3-2x^2+2x-2}{x^2+1}\,dx$

34. $\displaystyle\int \frac{x^4+6x^3+10x^2+x}{x^2+6x+10}\,dx$

35–38 True–False Determine whether the statement is true or false. Explain your answer.

35. The technique of partial fractions is used for integrals whose integrands are ratios of polynomials.

36. The integrand in

$$\int \frac{3x^4+5}{(x^2+1)^2}\,dx$$

is a proper rational function.

37. The partial fraction decomposition of

$$\frac{2x+3}{x^2} \quad \text{is} \quad \frac{2}{x}+\frac{3}{x^2}$$

38. If $f(x) = P(x)/(x+5)^3$ is a proper rational function, then the partial fraction decomposition of $f(x)$ has terms with constant numerators and denominators $(x+5)$, $(x+5)^2$, and $(x+5)^3$.

39–42 Evaluate the integral by making a substitution that converts the integrand to a rational function.

39. $\displaystyle\int \frac{\cos\theta}{\sin^2\theta+4\sin\theta-5}\,d\theta$

40. $\displaystyle\int \frac{e^t}{e^{2t}-4}\,dt$

41. $\displaystyle\int \frac{e^{3x}}{e^{2x}+4}\,dx$

42. $\displaystyle\int \frac{5+2\ln x}{x(1+\ln x)^2}\,dx$

43. Find the volume of the solid generated when the region enclosed by $y = x^2/(9-x^2)$, $y=0$, $x=0$, and $x=2$ is revolved about the x-axis.

44. Find the area of the region under the curve $y = 1/(1+e^x)$, over the interval $[-\ln 5, \ln 5]$. [*Hint:* Make a substitution that converts the integrand to a rational function.]

C **45–46** Use a CAS to evaluate the integral in two ways: (i) integrate directly; (ii) use the CAS to find the partial fraction decomposition and integrate the decomposition. Integrate by hand to check the results.

45. $\displaystyle\int \frac{x^2+1}{(x^2+2x+3)^2}\,dx$

46. $\displaystyle\int \frac{x^5+x^4+4x^3+4x^2+4x+4}{(x^2+2)^3}\,dx$

C **47–48** Integrate by hand and check your answers using a CAS.

47. $\displaystyle\int \frac{dx}{x^4-3x^3-7x^2+27x-18}$

48. $\displaystyle\int \frac{dx}{16x^3-4x^2+4x-1}$

FOCUS ON CONCEPTS

49. Show that

$$\int_0^1 \frac{x}{x^4+1}\,dx = \frac{\pi}{8}$$

50. Use partial fractions to derive the integration formula

$$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$

51. Suppose that ax^2+bx+c is a quadratic polynomial and that the integration

$$\int \frac{1}{ax^2+bx+c}\,dx$$

produces a function with no inverse tangent terms. What does this tell you about the roots of the polynomial?

52. Suppose that ax^2+bx+c is a quadratic polynomial and that the integration

$$\int \frac{1}{ax^2+bx+c}\,dx$$

produces a function with neither logarithmic nor inverse tangent terms. What does this tell you about the roots of the polynomial?

53. Does there exist a quadratic polynomial ax^2+bx+c such that the integration

$$\int \frac{x}{ax^2+bx+c}\,dx$$

produces a function with no logarithmic terms? If so, give an example; if not, explain why no such polynomial can exist.

54. **Writing** Suppose that $P(x)$ is a cubic polynomial. State the general form of the partial fraction decomposition for

$$f(x) = \frac{P(x)}{(x+5)^4}$$

and state the implications of this decomposition for evaluating the integral $\int f(x)\,dx$.

55. **Writing** Consider the functions

$$f(x) = \frac{1}{x^2-4} \quad \text{and} \quad g(x) = \frac{x}{x^2-4}$$

Each of the integrals $\int f(x)\,dx$ and $\int g(x)\,dx$ can be evaluated using partial fractions and using at least one other integration technique. Demonstrate two different techniques for evaluating each of these integrals, and then discuss the considerations that would determine which technique you would use.

QUICK CHECK EXERCISES 7.8 (See page 557 for answers.)

1. In each part, determine whether the integral is improper, and if so, explain why. Do not evaluate the integrals.
(a) $\int_{\pi/4}^{3\pi/4} \cot x \, dx$ (b) $\int_{\pi/4}^{\pi} \cot x \, dx$
(c) $\int_{0}^{+\infty} \frac{1}{x^2+1} \, dx$ (d) $\int_{1}^{+\infty} \frac{1}{x^2-1} \, dx$

2. Express each improper integral in Quick Check Exercise 1 in terms of one or more appropriate limits. Do not evaluate the limits.

3. The improper integral
$$\int_1^{+\infty} x^{-p} \, dx$$
converges to ________ provided ________.

4. Evaluate the integrals that converge.
(a) $\int_0^{+\infty} e^{-x} \, dx$ (b) $\int_0^{+\infty} e^{x} \, dx$
(c) $\int_0^{1} \frac{1}{x^3} \, dx$ (d) $\int_0^{1} \frac{1}{\sqrt[3]{x^2}} \, dx$

EXERCISE SET 7.8 Graphing Utility C CAS

1. In each part, determine whether the integral is improper, and if so, explain why.
(a) $\int_1^5 \frac{dx}{x-3}$ (b) $\int_1^5 \frac{dx}{x+3}$ (c) $\int_0^1 \ln x \, dx$
(d) $\int_1^{+\infty} e^{-x} \, dx$ (e) $\int_{-\infty}^{+\infty} \frac{dx}{\sqrt[3]{x-1}}$ (f) $\int_0^{\pi/4} \tan x \, dx$

2. In each part, determine all values of p for which the integral is improper.
(a) $\int_0^1 \frac{dx}{x^p}$ (b) $\int_1^2 \frac{dx}{x-p}$ (c) $\int_0^1 e^{-px} \, dx$

3–32 Evaluate the integrals that converge.

3. $\int_0^{+\infty} e^{-2x} \, dx$
4. $\int_{-1}^{+\infty} \frac{x}{1+x^2} \, dx$
5. $\int_3^{+\infty} \frac{2}{x^2-1} \, dx$
6. $\int_0^{+\infty} xe^{-x^2} \, dx$
7. $\int_e^{+\infty} \frac{1}{x \ln^3 x} \, dx$
8. $\int_2^{+\infty} \frac{1}{x\sqrt{\ln x}} \, dx$
9. $\int_{-\infty}^{0} \frac{dx}{(2x-1)^3}$
10. $\int_{-\infty}^{3} \frac{dx}{x^2+9}$
11. $\int_{-\infty}^{0} e^{3x} \, dx$
12. $\int_{-\infty}^{0} \frac{e^x \, dx}{3-2e^x}$
13. $\int_{-\infty}^{+\infty} x \, dx$
14. $\int_{-\infty}^{+\infty} \frac{x}{\sqrt{x^2+2}} \, dx$
15. $\int_{-\infty}^{+\infty} \frac{x}{(x^2+3)^2} \, dx$
16. $\int_{-\infty}^{+\infty} \frac{e^{-t}}{1+e^{-2t}} \, dt$
17. $\int_0^4 \frac{dx}{(x-4)^2}$
18. $\int_0^8 \frac{dx}{\sqrt[3]{x}}$
19. $\int_0^{\pi/2} \tan x \, dx$
20. $\int_0^4 \frac{dx}{\sqrt{4-x}}$
21. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$
22. $\int_{-3}^{1} \frac{x \, dx}{\sqrt{9-x^2}}$
23. $\int_{\pi/3}^{\pi/2} \frac{\sin x}{\sqrt{1-2\cos x}} \, dx$
24. $\int_0^{\pi/4} \frac{\sec^2 x}{1-\tan x} \, dx$
25. $\int_0^3 \frac{dx}{x-2}$
26. $\int_{-2}^{2} \frac{dx}{x^2}$
27. $\int_{-1}^{8} x^{-1/3} \, dx$
28. $\int_0^1 \frac{dx}{(x-1)^{2/3}}$
29. $\int_0^{+\infty} \frac{1}{x^2} \, dx$
30. $\int_1^{+\infty} \frac{dx}{x\sqrt{x^2-1}}$
31. $\int_0^1 \frac{dx}{\sqrt{x}(x+1)}$
32. $\int_0^{+\infty} \frac{dx}{\sqrt{x}(x+1)}$

33–36 True–False Determine whether the statement is true or false. Explain your answer.

33. $\int_1^{+\infty} x^{-4/3} \, dx$ converges to 3.

34. If f is continuous on $[a, +\infty)$ and $\lim_{x\to+\infty} f(x) = 1$, then $\int_a^{+\infty} f(x) \, dx$ converges.

35. $\int_1^2 \frac{1}{x(x-3)} \, dx$ is an improper integral.

36. $\int_{-1}^{1} \frac{1}{x^3} \, dx = 0$

37–40 Make the u-substitution and evaluate the resulting definite integral.

37. $\int_0^{+\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}} \, dx$; $u = \sqrt{x}$ [*Note:* $u \to +\infty$ as $x \to +\infty$.]

38. $\int_{12}^{+\infty} \frac{dx}{\sqrt{x}(x+4)}$; $u = \sqrt{x}$ [*Note:* $u \to +\infty$ as $x \to +\infty$.]

39. $\int_0^{+\infty} \frac{e^{-x}}{\sqrt{1-e^{-x}}} \, dx$; $u = 1 - e^{-x}$
[*Note:* $u \to 1$ as $x \to +\infty$.]

40. $\displaystyle\int_0^{+\infty} \frac{e^{-x}}{\sqrt{1-e^{-2x}}}\,dx;\ u = e^{-x}$

C **41–42** Express the improper integral as a limit, and then evaluate that limit with a CAS. Confirm the answer by evaluating the integral directly with the CAS. ■

41. $\displaystyle\int_0^{+\infty} e^{-x}\cos x\,dx$ **42.** $\displaystyle\int_0^{+\infty} xe^{-3x}\,dx$

C **43.** In each part, try to evaluate the integral exactly with a CAS. If your result is not a simple numerical answer, then use the CAS to find a numerical approximation of the integral.

(a) $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{x^8+x+1}\,dx$ (b) $\displaystyle\int_0^{+\infty} \frac{1}{\sqrt{1+x^3}}\,dx$

(c) $\displaystyle\int_1^{+\infty} \frac{\ln x}{e^x}\,dx$ (d) $\displaystyle\int_1^{+\infty} \frac{\sin x}{x^2}\,dx$

C **44.** In each part, confirm the result with a CAS.

(a) $\displaystyle\int_0^{+\infty} \frac{\sin x}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2}}$ (b) $\displaystyle\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}$

(c) $\displaystyle\int_0^{1} \frac{\ln x}{1+x}\,dx = -\frac{\pi^2}{12}$

45. Find the length of the curve $y = (4 - x^{2/3})^{3/2}$ over the interval [0, 8].

46. Find the length of the curve $y = \sqrt{4-x^2}$ over the interval [0, 2].

47–48 Use L'Hôpital's rule to help evaluate the improper integral. ■

47. $\displaystyle\int_0^{1} \ln x\,dx$ **48.** $\displaystyle\int_1^{+\infty} \frac{\ln x}{x^2}\,dx$

49. Find the area of the region between the x-axis and the curve $y = e^{-3x}$ for $x \ge 0$.

50. Find the area of the region between the x-axis and the curve $y = 8/(x^2-4)$ for $x \ge 4$.

51. Suppose that the region between the x-axis and the curve $y = e^{-x}$ for $x \ge 0$ is revolved about the x-axis.

(a) Find the volume of the solid that is generated.

(b) Find the surface area of the solid.

FOCUS ON CONCEPTS

52. Suppose that f and g are continuous functions and that

$$0 \le f(x) \le g(x)$$

if $x \ge a$. Give a reasonable informal argument using areas to explain why the following results are true.

(a) If $\int_a^{+\infty} f(x)\,dx$ diverges, then $\int_a^{+\infty} g(x)\,dx$ diverges.

(b) If $\int_a^{+\infty} g(x)\,dx$ converges, then $\int_a^{+\infty} f(x)\,dx$ converges and $\int_a^{+\infty} f(x)\,dx \le \int_a^{+\infty} g(x)\,dx$.

[*Note:* The results in this exercise are sometimes called ***comparison tests*** for improper integrals.]

53–56 Use the results in Exercise 52. ■

53. (a) Confirm graphically and algebraically that

$$e^{-x^2} \le e^{-x} \quad (x \ge 1)$$

(b) Evaluate the integral

$$\int_1^{+\infty} e^{-x}\,dx$$

(c) What does the result obtained in part (b) tell you about the integral

$$\int_1^{+\infty} e^{-x^2}\,dx?$$

54. (a) Confirm graphically and algebraically that

$$\frac{1}{2x+1} \le \frac{e^x}{2x+1} \quad (x \ge 0)$$

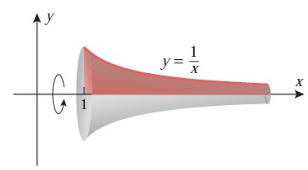
(b) Evaluate the integral

$$\int_0^{+\infty} \frac{dx}{2x+1}$$

(c) What does the result obtained in part (b) tell you about the integral

$$\int_0^{+\infty} \frac{e^x}{2x+1}\,dx?$$

55. Let R be the region to the right of $x = 1$ that is bounded by the x-axis and the curve $y = 1/x$. When this region is revolved about the x-axis it generates a solid whose surface is known as ***Gabriel's Horn*** (for reasons that should be clear from the accompanying figure). Show that the solid has a finite volume but its surface has an infinite area. [*Note:* It has been suggested that if one could saturate the interior of the solid with paint and allow it to seep through to the surface, then one could paint an infinite surface with a finite amount of paint! What do you think?]



◀ **Figure Ex-55**

56. In each part, use Exercise 52 to determine whether the integral converges or diverges. If it converges, then use part (b) of that exercise to find an upper bound on the value of the integral.

(a) $\displaystyle\int_2^{+\infty} \frac{\sqrt{x^3+1}}{x}\,dx$ (b) $\displaystyle\int_2^{+\infty} \frac{x}{x^5+1}\,dx$

(c) $\displaystyle\int_0^{+\infty} \frac{xe^x}{2x+1}\,dx$

QUICK CHECK EXERCISES 6.1 *(See page 421 for answers.)*

1. An integral expression for the area of the region between the curves $y = 20 - 3x^2$ and $y = e^x$ and bounded on the sides by $x = 0$ and $x = 2$ is ________.

2. An integral expression for the area of the parallelogram bounded by $y = 2x + 8$, $y = 2x - 3$, $x = -1$, and $x = 5$ is ________. The value of this integral is ________.

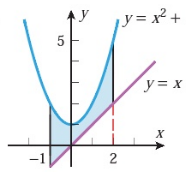
3. (a) The points of intersection for the circle $x^2 + y^2 = 4$ and the line $y = x + 2$ are ________ and ________.
   (b) Expressed as a definite integral with respect to x, ________ gives the area of the region inside the circle $x^2 + y^2 = 4$ and above the line $y = x + 2$.
   (c) Expressed as a definite integral with respect to y, ________ gives the area of the region described in part (b).

4. The area of the region enclosed by the curves $y = x^2$ and $y = \sqrt[3]{x}$ is ________.

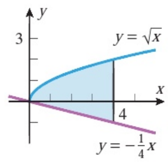
EXERCISE SET 6.1 Graphing Utility C CAS

1–4 Find the area of the shaded region.

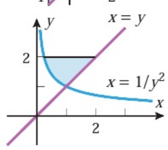
1.



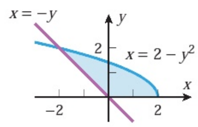
2.



3.

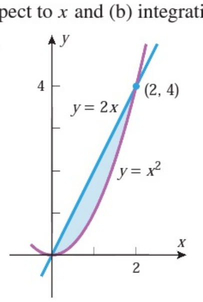


4.

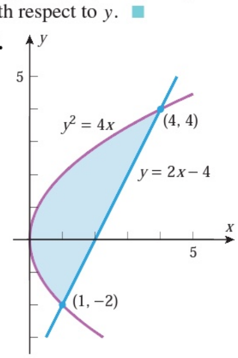


5–6 Find the area of the shaded region by (a) integrating with respect to x and (b) integrating with respect to y.

5.



6.



7–18 Sketch the region enclosed by the curves and find its area.

7. $y = x^2, \; y = \sqrt{x}, \; x = \frac{1}{4}, \; x = 1$

8. $y = x^3 - 4x, \; y = 0, \; x = 0, \; x = 2$

9. $y = \cos 2x, \; y = 0, \; x = \pi/4, \; x = \pi/2$

10. $y = \sec^2 x, \; y = 2, \; x = -\pi/4, \; x = \pi/4$

11. $x = \sin y, \; x = 0, \; y = \pi/4, \; y = 3\pi/4$

12. $x^2 = y, \; x = y - 2$

13. $y = e^x, \; y = e^{2x}, \; x = 0, \; x = \ln 2$

14. $x = 1/y, \; x = 0, \; y = 1, \; y = e$

15. $y = \dfrac{2}{1 + x^2}, \; y = |x|$

16. $y = \dfrac{1}{\sqrt{1 - x^2}}, \; y = 2$

17. $y = 2 + |x - 1|, \; y = -\frac{1}{5}x + 7$

18. $y = x, \; y = 4x, \; y = -x + 2$

19–26 Use a graphing utility, where helpful, to find the area of the region enclosed by the curves.

19. $y = x^3 - 4x^2 + 3x, \; y = 0$

20. $y = x^3 - 2x^2, \; y = 2x^2 - 3x$

21. $y = \sin x, \; y = \cos x, \; x = 0, \; x = 2\pi$

22. $y = x^3 - 4x, \; y = 0$

23. $x = y^3 - y, \; x = 0$

24. $x = y^3 - 4y^2 + 3y, \; x = y^2 - y$

25. $y = xe^{x^2}, \; y = 2|x|$

26. $y = \dfrac{1}{x\sqrt{1 - (\ln x)^2}}, \; y = \dfrac{3}{x}$

27–30 True–False Determine whether the statement is true or false. Explain your answer. [In each exercise, assume that f and g are distinct continuous functions on $[a, b]$ and that A denotes the area of the region bounded by the graphs of $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$.]

27. If f and g differ by a positive constant c, then $A = c(b - a)$.

28. If
$$\int_a^b [f(x) - g(x)]\, dx = -3$$
then $A = 3$.

29. If
$$\int_a^b [f(x) - g(x)]\, dx = 0$$
then the graphs of $y = f(x)$ and $y = g(x)$ cross at least once on $[a, b]$.

30. If
$$A = \left| \int_a^b [f(x) - g(x)]\, dx \right|$$
then the graphs of $y = f(x)$ and $y = g(x)$ don't cross on $[a, b]$.

QUICK CHECK EXERCISES 6.2 *(See page 431 for answers.)* วิธีแผ่นกลม

1. A solid S extends along the x-axis from $x = 1$ to $x = 3$. For x between 1 and 3, the cross-sectional area of S perpendicular to the x-axis is $3x^2$. An integral expression for the volume of S is ________. The value of this integral is ________.

2. A solid S is generated by revolving the region between the x-axis and the curve $y = \sqrt{\sin x}$ $(0 \le x \le \pi)$ about the x-axis.
   (a) For x between 0 and π, the cross-sectional area of S perpendicular to the x-axis at x is $A(x) =$ ________.
   (b) An integral expression for the volume of S is ________.
   (c) The value of the integral in part (b) is ________.

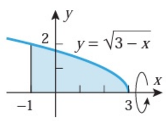
3. A solid S is generated by revolving the region enclosed by the line $y = 2x + 1$ and the curve $y = x^2 + 1$ about the x-axis.
   (a) For x between ________ and ________, the cross-sectional area of S perpendicular to the x-axis at x is $A(x) =$ ________.
   (b) An integral expression for the volume of S is ________.

4. A solid S is generated by revolving the region enclosed by the line $y = x + 1$ and the curve $y = x^2 + 1$ about the y-axis.
   (a) For y between ________ and ________, the cross-sectional area of S perpendicular to the y-axis at y is $A(y) =$ ________.
   (b) An integral expression for the volume of S is ________.

EXERCISE SET 6.2 C CAS

1–8 Find the volume of the solid that results when the shaded region is revolved about the indicated axis. ■

1.



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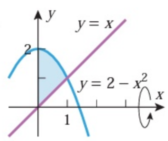

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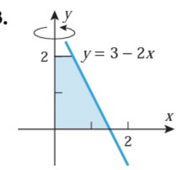

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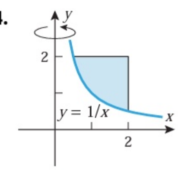

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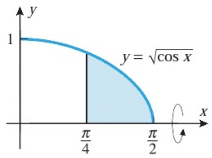

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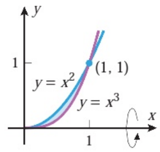

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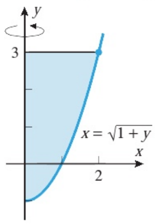

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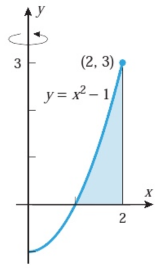


9. Find the volume of the solid whose base is the region bounded between the curve $y = x^2$ and the x-axis from $x = 0$ to $x = 2$ and whose cross sections taken perpendicular to the x-axis are squares.

10. Find the volume of the solid whose base is the region bounded between the curve $y = \sec x$ and the x-axis from $x = \pi/4$ to $x = \pi/3$ and whose cross sections taken perpendicular to the x-axis are squares.

11–18 Find the volume of the solid that results when the region enclosed by the given curves is revolved about the x-axis. ■

11. $y = \sqrt{25 - x^2},\ \ y = 3$

12. $y = 9 - x^2,\ \ y = 0$ **13.** $x = \sqrt{y},\ \ x = y/4$

14. $y = \sin x,\ \ y = \cos x,\ \ x = 0,\ \ x = \pi/4$
[*Hint:* Use the identity $\cos 2x = \cos^2 x - \sin^2 x$.]

15. $y = e^x,\ \ y = 0,\ \ x = 0,\ \ x = \ln 3$

16. $y = e^{-2x},\ \ y = 0,\ \ x = 0,\ \ x = 1$

17. $y = \dfrac{1}{\sqrt{4 + x^2}},\ \ x = -2,\ \ x = 2,\ \ y = 0$

18. $y = \dfrac{e^{3x}}{\sqrt{1 + e^{6x}}},\ \ x = 0,\ \ x = 1,\ \ y = 0$

19. Find the volume of the solid whose base is the region bounded between the curve $y = x^3$ and the y-axis from $y = 0$ to $y = 1$ and whose cross sections taken perpendicular to the y-axis are squares.

20. Find the volume of the solid whose base is the region enclosed between the curve $x = 1 - y^2$ and the y-axis and whose cross sections taken perpendicular to the y-axis are squares.

21–26 Find the volume of the solid that results when the region enclosed by the given curves is revolved about the y-axis. ■

21. $x = \csc y,\ \ y = \pi/4,\ \ y = 3\pi/4,\ \ x = 0$

22. $y = x^2,\ \ x = y^2$

23. $x = y^2,\ \ x = y + 2$

24. $x = 1 - y^2,\ \ x = 2 + y^2,\ \ y = -1,\ \ y = 1$

25. $y = \ln x,\ \ x = 0,\ \ y = 0,\ \ y = 1$

26. $y = \sqrt{\dfrac{1 - x^2}{x^2}}\quad (x > 0),\ \ x = 0,\ \ y = 0,\ \ y = 2$

27–30 True–False Determine whether the statement is true or false. Explain your answer. [In these exercises, assume that a solid S of volume V is bounded by two parallel planes perpendicular to the x-axis at $x = a$ and $x = b$ and that for each x in $[a, b]$, $A(x)$ denotes the cross-sectional area of S perpendicular to the x-axis.] ■

27. If each cross section of S perpendicular to the x-axis is a square, then S is a rectangular parallelepiped (i.e., is box shaped).

28. If each cross section of S is a disk or a washer, then S is a solid of revolution.

29. If x is in centimeters (cm), then $A(x)$ must be a quadratic function of x, since units of $A(x)$ will be square centimeters (cm^2).

30. The average value of $A(x)$ on the interval $[a, b]$ is given by $V/(b - a)$.

31. Find the volume of the solid that results when the region above the x-axis and below the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > 0, b > 0)$$

is revolved about the x-axis.

32. Let V be the volume of the solid that results when the region enclosed by $y = 1/x$, $y = 0$, $x = 2$, and $x = b$ $(0 < b < 2)$ is revolved about the x-axis. Find the value of b for which $V = 3$.

33. Find the volume of the solid generated when the region enclosed by $y = \sqrt{x + 1}$, $y = \sqrt{2x}$, and $y = 0$ is revolved about the x-axis. [*Hint:* Split the solid into two parts.]

34. Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 6 - x$, and $y = 0$ is revolved about the x-axis. [*Hint:* Split the solid into two parts.]

FOCUS ON CONCEPTS

35. Suppose that f is a continuous function on $[a, b]$, and let R be the region between the curve $y = f(x)$ and the line $y = k$ from $x = a$ to $x = b$. Using the method of disks, derive with explanation a formula for the volume of a solid generated by revolving R about the line $y = k$. State and explain additional assumptions, if any, that you need about f for your formula.

36. Suppose that v and w are continuous functions on $[c, d]$, and let R be the region between the curves $x = v(y)$ and $x = w(y)$ from $y = c$ to $y = d$. Using the method of washers, derive with explanation a formula for the volume of a solid generated by revolving R about the line

QUICK CHECK EXERCISES 6.3 (See page 438 for answers.) วิธีเปลือกทรงกระบอก

1. Let R be the region between the x-axis and the curve $y = 1 + \sqrt{x}$ for $1 \le x \le 4$.
   (a) For x between 1 and 4, the area of the cylindrical surface generated by revolving the vertical cross section of R at x about the y-axis is ________.
   (b) Using cylindrical shells, an integral expression for the volume of the solid generated by revolving R about the y-axis is ________.

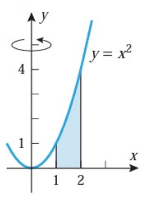
2. Let R be the region described in Quick Check Exercise 1.
   (a) For x between 1 and 4, the area of the cylindrical surface generated by revolving the vertical cross section of R at x about the line $x = 5$ is ________.
   (b) Using cylindrical shells, an integral expression for the volume of the solid generated by revolving R about the line $x = 5$ is ________.

3. A solid S is generated by revolving the region enclosed by the curves $x = (y-2)^2$ and $x = 4$ about the x-axis. Using cylindrical shells, an integral expression for the volume of S is ________.

EXERCISE SET 6.3 C CAS

1–4 Use cylindrical shells to find the volume of the solid generated when the shaded region is revolved about the indicated axis.

1.



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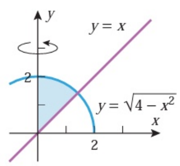

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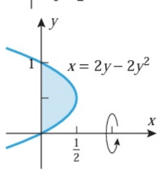

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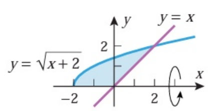


5–12 Use cylindrical shells to find the volume of the solid generated when the region enclosed by the given curves is revolved about the y-axis.

5. $y = x^3,\ x = 1,\ y = 0$
6. $y = \sqrt{x},\ x = 4,\ x = 9,\ y = 0$
7. $y = 1/x,\ y = 0,\ x = 1,\ x = 3$
8. $y = \cos(x^2),\ x = 0,\ x = \frac{1}{2}\sqrt{\pi},\ y = 0$
9. $y = 2x - 1,\ y = -2x + 3,\ x = 2$
10. $y = 2x - x^2,\ y = 0$
11. $y = \dfrac{1}{x^2 + 1},\ x = 0,\ x = 1,\ y = 0$
12. $y = e^{x^2},\ x = 1,\ x = \sqrt{3},\ y = 0$

13–16 Use cylindrical shells to find the volume of the solid generated when the region enclosed by the given curves is revolved about the x-axis.

13. $y^2 = x,\ \ y = 1,\ \ x = 0$

14. $x = 2y,\ \ y = 2,\ \ y = 3,\ \ x = 0$

15. $y = x^2,\ \ x = 1,\ \ y = 0$ **16.** $xy = 4,\ \ x + y = 5$

17–20 True–False Determine whether the statement is true or false. Explain your answer. ■

17. The volume of a cylindrical shell is equal to the product of the thickness of the shell with the surface area of a cylinder whose height is that of the shell and whose radius is equal to the average of the inner and outer radii of the shell.

18. The method of cylindrical shells is a special case of the method of integration of cross-sectional area that was discussed in Section 6.2.

19. In the method of cylindrical shells, integration is over an interval on a coordinate axis that is *perpendicular* to the axis of revolution of the solid.

20. The Riemann sum approximation

$$V \approx \sum_{k=1}^{n} 2\pi x_k^* f(x_k^*)\Delta x_k \qquad \left(\text{where } x_k^* = \frac{x_k + x_{k-1}}{2}\right)$$

for the volume of a solid of revolution is exact when f is a constant function.

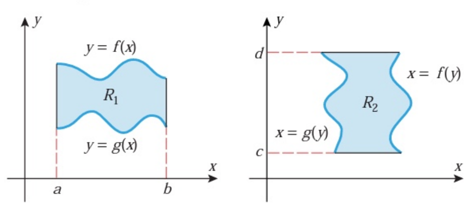
C **21.** Use a CAS to find the volume of the solid generated when the region enclosed by $y = e^x$ and $y = 0$ for $1 \le x \le 2$ is revolved about the y-axis.

C **22.** Use a CAS to find the volume of the solid generated when the region enclosed by $y = \cos x$, $y = 0$, and $x = 0$ for $0 \le x \le \pi/2$ is revolved about the y-axis.

C **23.** Consider the region to the right of the y-axis, to the left of the vertical line $x = k$ $(0 < k < \pi)$, and between the curve $y = \sin x$ and the x-axis. Use a CAS to estimate the value of k so that the solid generated by revolving the region about the y-axis has a volume of 8 cubic units.

FOCUS ON CONCEPTS

24. Let R_1 and R_2 be regions of the form shown in the accompanying figure. Use cylindrical shells to find a formula for the volume of the solid that results when
(a) region R_1 is revolved about the y-axis
(b) region R_2 is revolved about the x-axis.



▲ **Figure Ex-24**

25. (a) Use cylindrical shells to find the volume of the solid that is generated when the region under the curve

$$y = x^3 - 3x^2 + 2x$$

over $[0, 1]$ is revolved about the y-axis.
(b) For this problem, is the method of cylindrical shells easier or harder than the method of slicing discussed in the last section? Explain.

26. Let f be continuous and nonnegative on $[a, b]$, and let R be the region that is enclosed by $y = f(x)$ and $y = 0$ for $a \le x \le b$. Using the method of cylindrical shells, derive with explanation a formula for the volume of the solid generated by revolving R about the line $x = k$, where $k \le a$.

27–28 Using the method of cylindrical shells, set up but do not evaluate an integral for the volume of the solid generated when the region R is revolved about (a) the line $x = 1$ and (b) the line $y = -1$. ■

27. R is the region bounded by the graphs of $y = x$, $y = 0$, and $x = 1$.

28. R is the region in the first quadrant bounded by the graphs of $y = \sqrt{1 - x^2}$, $y = 0$, and $x = 0$.

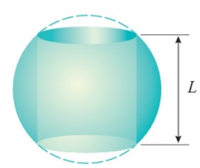
29. Use cylindrical shells to find the volume of the solid that is generated when the region that is enclosed by $y = 1/x^3$, $x = 1$, $x = 2$, $y = 0$ is revolved about the line $x = -1$.

30. Use cylindrical shells to find the volume of the solid that is generated when the region that is enclosed by $y = x^3$, $y = 1$, $x = 0$ is revolved about the line $y = 1$.

31. Use cylindrical shells to find the volume of the cone generated when the triangle with vertices $(0, 0)$, $(0, r)$, $(h, 0)$, where $r > 0$ and $h > 0$, is revolved about the x-axis.

32. The region enclosed between the curve $y^2 = kx$ and the line $x = \frac{1}{4}k$ is revolved about the line $x = \frac{1}{2}k$. Use cylindrical shells to find the volume of the resulting solid. (Assume $k > 0$.)

33. As shown in the accompanying figure, a cylindrical hole is drilled all the way through the center of a sphere. Show that the volume of the remaining solid depends only on the length L of the hole, not on the size of the sphere.



◀ **Figure Ex-33**

34. Use cylindrical shells to find the volume of the torus obtained by revolving the circle $x^2 + y^2 = a^2$ about the line