แบบฝึกหัดเพิ่มเติม

EXERCISES 8.1

Basic Substitutions

Evaluate each integral in Exercises 1–36 by using a substitution to reduce it to standard form.

1. $\displaystyle\int \frac{16x\,dx}{\sqrt{8x^2+1}}$ **2.** $\displaystyle\int \frac{3\cos x\,dx}{\sqrt{1+3\sin x}}$

3. $\displaystyle\int 3\sqrt{\sin v}\cos v\,dv$ **4.** $\displaystyle\int \cot^3 y\csc^2 y\,dy$

5. $\displaystyle\int_0^1 \frac{16x\,dx}{8x^2+2}$ **6.** $\displaystyle\int_{\pi/4}^{\pi/3} \frac{\sec^2 z}{\tan z}\,dz$

7. $\displaystyle\int \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$ **8.** $\displaystyle\int \frac{dx}{x-\sqrt{x}}$

9. $\displaystyle\int \cot(3-7x)\,dx$ **10.** $\displaystyle\int \csc(\pi x-1)\,dx$

11. $\displaystyle\int e^\theta\csc(e^\theta+1)\,d\theta$ **12.** $\displaystyle\int \frac{\cot(3+\ln x)}{x}\,dx$

13. $\displaystyle\int \sec\frac{t}{3}\,dt$ **14.** $\displaystyle\int x\sec(x^2-5)\,dx$

15. $\displaystyle\int \csc(s-\pi)\,ds$ **16.** $\displaystyle\int \frac{1}{\theta^2}\csc\frac{1}{\theta}\,d\theta$

17. $\displaystyle\int_0^{\sqrt{\ln 2}} 2x\,e^{x^2}\,dx$ **18.** $\displaystyle\int_{\pi/2}^{\pi} (\sin y)e^{\cos y}\,dy$

19. $\displaystyle\int e^{\tan v}\sec^2 v\,dv$ **20.** $\displaystyle\int \frac{e^{\sqrt{t}}\,dt}{\sqrt{t}}$

21. $\displaystyle\int 3^{x+1}\,dx$ **22.** $\displaystyle\int \frac{2^{\ln x}}{x}\,dx$

23. $\displaystyle\int \frac{2^{\sqrt{w}}\,dw}{2\sqrt{w}}$ **24.** $\displaystyle\int 10^{2\theta}\,d\theta$

25. $\displaystyle\int \frac{9\,du}{1+9u^2}$ **26.** $\displaystyle\int \frac{4\,dx}{1+(2x+1)^2}$

27. $\displaystyle\int_0^{1/6} \frac{dx}{\sqrt{1-9x^2}}$ **28.** $\displaystyle\int_0^1 \frac{dt}{\sqrt{4-t^2}}$

29. $\displaystyle\int \frac{2s\,ds}{\sqrt{1-s^4}}$ **30.** $\displaystyle\int \frac{2\,dx}{x\sqrt{1-4\ln^2 x}}$

31. $\displaystyle\int \frac{6\,dx}{x\sqrt{25x^2-1}}$ **32.** $\displaystyle\int \frac{dr}{r\sqrt{r^2-9}}$

33. $\displaystyle\int \frac{dx}{e^x+e^{-x}}$ **34.** $\displaystyle\int \frac{dy}{\sqrt{e^{2y}-1}}$

35. $\displaystyle\int_1^{e^{\pi/3}} \frac{dx}{x\cos(\ln x)}$ **36.** $\displaystyle\int \frac{\ln x\,dx}{x+4x\ln^2 x}$

Completing the Square

Evaluate each integral in Exercises 37–42 by completing the square and using a substitution to reduce it to standard form.

37. $\displaystyle\int_1^2 \frac{8\,dx}{x^2-2x+2}$ **38.** $\displaystyle\int_2^4 \frac{2\,dx}{x^2-6x+10}$

39. $\displaystyle\int \frac{dt}{\sqrt{-t^2+4t-3}}$ **40.** $\displaystyle\int \frac{d\theta}{\sqrt{2\theta-\theta^2}}$

41. $\displaystyle\int \frac{dx}{(x+1)\sqrt{x^2+2x}}$ **42.** $\displaystyle\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$

Trigonometric Identities

Evaluate each integral in Exercises 43–46 by using trigonometric identities and substitutions to reduce it to standard form.

43. $\displaystyle\int (\sec x+\cot x)^2\,dx$ **44.** $\displaystyle\int (\csc x-\tan x)^2\,dx$

45. $\displaystyle\int \csc x\sin 3x\,dx$

46. $\displaystyle\int (\sin 3x\cos 2x-\cos 3x\sin 2x)\,dx$

Improper Fractions

Evaluate each integral in Exercises 47–52 by reducing the improper fraction and using a substitution (if necessary) to reduce it to standard form.

47. $\displaystyle\int \frac{x}{x+1}\,dx$ **48.** $\displaystyle\int \frac{x^2}{x^2+1}\,dx$

49. $\displaystyle\int_{\sqrt{2}}^{3} \frac{2x^3}{x^2-1}\,dx$ **50.** $\displaystyle\int_{-1}^{3} \frac{4x^2-7}{2x+3}\,dx$

51. $\displaystyle\int \frac{4t^3-t^2+16t}{t^2+4}\,dt$ **52.** $\displaystyle\int \frac{2\theta^3-7\theta^2+7\theta}{2\theta-5}\,d\theta$

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Separating Fractions

Evaluate each integral in Exercises 53–56 by separating the fraction and using a substitution (if necessary) to reduce it to standard form.

53. $\displaystyle\int \frac{1-x}{\sqrt{1-x^2}}\,dx$ **54.** $\displaystyle\int \frac{x+2\sqrt{x-1}}{2x\sqrt{x-1}}\,dx$

55. $\displaystyle\int_0^{\pi/4} \frac{1+\sin x}{\cos^2 x}\,dx$ **56.** $\displaystyle\int_0^{1/2} \frac{2-8x}{1+4x^2}\,dx$

Multiplying by a Form of 1

Evaluate each integral in Exercises 57–62 by multiplying by a form of 1 and using a substitution (if necessary) to reduce it to standard form.

57. $\displaystyle\int \frac{1}{1+\sin x}\,dx$ **58.** $\displaystyle\int \frac{1}{1+\cos x}\,dx$

59. $\displaystyle\int \frac{1}{\sec\theta+\tan\theta}\,d\theta$ **60.** $\displaystyle\int \frac{1}{\csc\theta+\cot\theta}\,d\theta$

61. $\displaystyle\int \frac{1}{1-\sec x}\,dx$ **62.** $\displaystyle\int \frac{1}{1-\csc x}\,dx$

Eliminating Square Roots

Evaluate each integral in Exercises 63–70 by eliminating the square root.

63. $\displaystyle\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}}\,dx$ **64.** $\displaystyle\int_0^{\pi} \sqrt{1-\cos 2x}\,dx$

65. $\int_{\pi/2}^{\pi} \sqrt{1 + \cos 2t}\, dt$

66. $\int_{-\pi}^{0} \sqrt{1 + \cos t}\, dt$

67. $\int_{-\pi}^{0} \sqrt{1 - \cos^2 \theta}\, d\theta$

68. $\int_{\pi/2}^{\pi} \sqrt{1 - \sin^2 \theta}\, d\theta$

69. $\int_{-\pi/4}^{\pi/4} \sqrt{1 + \tan^2 y}\, dy$

70. $\int_{-\pi/4}^{0} \sqrt{\sec^2 y - 1}\, dy$

Assorted Integrations

Evaluate each integral in Exercises 71–82 by using any technique you think is appropriate.

71. $\int_{\pi/4}^{3\pi/4} (\csc x - \cot x)^2\, dx$

72. $\int_{0}^{\pi/4} (\sec x + 4\cos x)^2\, dx$

73. $\int \cos\theta \csc(\sin\theta)\, d\theta$

74. $\int \left(1 + \frac{1}{x}\right) \cot(x + \ln x)\, dx$

75. $\int (\csc x - \sec x)(\sin x + \cos x)\, dx$

76. $\int 3\sinh\left(\frac{x}{2} + \ln 5\right) dx$

77. $\int \frac{6\, dy}{\sqrt{y}(1 + y)}$

78. $\int \frac{dx}{x\sqrt{4x^2 - 1}}$

79. $\int \frac{7\, dx}{(x - 1)\sqrt{x^2 - 2x - 48}}$

80. $\int \frac{dx}{(2x + 1)\sqrt{4x^2 + 4x}}$

81. $\int \sec^2 t \tan(\tan t)\, dt$

82. $\int \frac{dx}{x\sqrt{3 + x^2}}$

Trigonometric Powers

83. **a.** Evaluate $\int \cos^3 \theta\, d\theta$. (*Hint:* $\cos^2\theta = 1 - \sin^2\theta$.)
    b. Evaluate $\int \cos^5 \theta\, d\theta$.
    c. Without actually evaluating the integral, explain how you would evaluate $\int \cos^9 \theta\, d\theta$.

84. **a.** Evaluate $\int \sin^3 \theta\, d\theta$. (*Hint:* $\sin^2\theta = 1 - \cos^2\theta$.)
    b. Evaluate $\int \sin^5 \theta\, d\theta$.
    c. Evaluate $\int \sin^7 \theta\, d\theta$.
    d. Without actually evaluating the integral, explain how you would evaluate $\int \sin^{13} \theta\, d\theta$.

85. **a.** Express $\int \tan^3 \theta\, d\theta$ in terms of $\int \tan\theta\, d\theta$. Then evaluate $\int \tan^3\theta\, d\theta$. (*Hint:* $\tan^2\theta = \sec^2\theta - 1$.)
    b. Express $\int \tan^5 \theta\, d\theta$ in terms of $\int \tan^3 \theta\, d\theta$.
    c. Express $\int \tan^7 \theta\, d\theta$ in terms of $\int \tan^5 \theta\, d\theta$.
    d. Express $\int \tan^{2k+1} \theta\, d\theta$, where k is a positive integer, in terms of $\int \tan^{2k-1} \theta\, d\theta$.

86. **a.** Express $\int \cot^3 \theta\, d\theta$ in terms of $\int \cot\theta\, d\theta$. Then evaluate $\int \cot^3\theta\, d\theta$. (*Hint:* $\cot^2\theta = \csc^2\theta - 1$.)
    b. Express $\int \cot^5 \theta\, d\theta$ in terms of $\int \cot^3 \theta\, d\theta$.
    c. Express $\int \cot^7 \theta\, d\theta$ in terms of $\int \cot^5 \theta\, d\theta$.
    d. Express $\int \cot^{2k+1} \theta\, d\theta$, where k is a positive integer, in terms of $\int \cot^{2k-1} \theta\, d\theta$.

Theory and Examples

87. **Area** Find the area of the region bounded above by $y = 2\cos x$ and below by $y = \sec x$, $-\pi/4 \le x \le \pi/4$.

88. **Area** Find the area of the "triangular" region that is bounded from above and below by the curves $y = \csc x$ and $y = \sin x$, $\pi/6 \le x \le \pi/2$, and on the left by the line $x = \pi/6$.

89. **Volume** Find the volume of the solid generated by revolving the region in Exercise 87 about the x-axis.

90. **Volume** Find the volume of the solid generated by revolving the region in Exercise 88 about the x-axis.

91. **Arc length** Find the length of the curve $y = \ln(\cos x)$, $0 \le x \le \pi/3$.

92. **Arc length** Find the length of the curve $y = \ln(\sec x)$, $0 \le x \le \pi/4$.

93. **Centroid** Find the centroid of the region bounded by the x-axis, the curve $y = \sec x$, and the lines $x = -\pi/4$, $x = \pi/4$.

94. **Centroid** Find the centroid of the region that is bounded by the x-axis, the curve $y = \csc x$, and the lines $x = \pi/6$, $x = 5\pi/6$.

95. **The integral of csc x** Repeat the derivation in Example 7, using cofunctions, to show that

$$\int \csc x\, dx = -\ln|\csc x + \cot x| + C.$$

96. **Using different substitutions** Show that the integral

$$\int ((x^2 - 1)(x + 1))^{-2/3}\, dx$$

can be evaluated with any of the following substitutions.

a. $u = 1/(x + 1)$
b. $u = ((x - 1)/(x + 1))^k$ for $k = 1, 1/2, 1/3, -1/3, -2/3$, and -1
c. $u = \tan^{-1} x$
d. $u = \tan^{-1}\sqrt{x}$
e. $u = \tan^{-1}((x - 1)/2)$
f. $u = \cos^{-1} x$
g. $u = \cosh^{-1} x$

What is the value of the integral? (*Source:* "Problems and Solutions," *College Mathematics Journal*, Vol. 21, No. 5 (Nov. 1990), pp. 425–426.)

EXERCISES 8.2

Integration by Parts

Evaluate the integrals in Exercises 1–24.

1. $\int x \sin \frac{x}{2}\, dx$
2. $\int \theta \cos \pi\theta\, d\theta$
3. $\int t^2 \cos t\, dt$
4. $\int x^2 \sin x\, dx$
5. $\int_1^2 x \ln x\, dx$
6. $\int_1^e x^3 \ln x\, dx$
7. $\int \tan^{-1} y\, dy$
8. $\int \sin^{-1} y\, dy$
9. $\int x \sec^2 x\, dx$
10. $\int 4x \sec^2 2x\, dx$
11. $\int x^3 e^x\, dx$
12. $\int p^4 e^{-p}\, dp$
13. $\int (x^2 - 5x)e^x\, dx$
14. $\int (r^2 + r + 1)e^r\, dr$
15. $\int x^5 e^x\, dx$
16. $\int t^2 e^{4t}\, dt$
17. $\int_0^{\pi/2} \theta^2 \sin 2\theta\, d\theta$
18. $\int_0^{\pi/2} x^3 \cos 2x\, dx$
19. $\int_{2/\sqrt{3}}^2 t \sec^{-1} t\, dt$
20. $\int_0^{1/\sqrt{2}} 2x \sin^{-1}(x^2)\, dx$
21. $\int e^\theta \sin \theta\, d\theta$
22. $\int e^{-y} \cos y\, dy$
23. $\int e^{2x} \cos 3x\, dx$
24. $\int e^{-2x} \sin 2x\, dx$

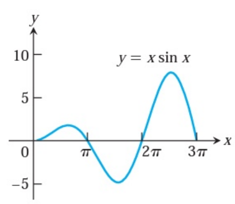
Substitution and Integration by Parts

Evaluate the integrals in Exercises 25–30 by using a substitution prior to integration by parts.

25. $\int e^{\sqrt{3s+9}}\, ds$
26. $\int_0^1 x\sqrt{1 - x}\, dx$
27. $\int_0^{\pi/3} x \tan^2 x\, dx$
28. $\int \ln(x + x^2)\, dx$
29. $\int \sin(\ln x)\, dx$
30. $\int z(\ln z)^2\, dz$

Theory and Examples

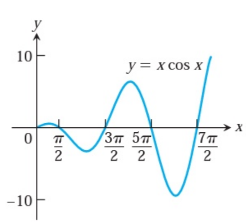
31. **Finding area** Find the area of the region enclosed by the curve $y = x \sin x$ and the x-axis (see the accompanying figure) for
   a. $0 \le x \le \pi$ **b.** $\pi \le x \le 2\pi$ **c.** $2\pi \le x \le 3\pi$.
   d. What pattern do you see here? What is the area between the curve and the x-axis for $n\pi \le x \le (n + 1)\pi$, n an arbitrary nonnegative integer? Give reasons for your answer.



32. **Finding area** Find the area of the region enclosed by the curve $y = x \cos x$ and the x-axis (see the accompanying figure) for
   a. $\pi/2 \le x \le 3\pi/2$ **b.** $3\pi/2 \le x \le 5\pi/2$
   c. $5\pi/2 \le x \le 7\pi/2$.
   d. What pattern do you see? What is the area between the curve and the x-axis for

$$\left(\frac{2n-1}{2}\right)\pi \le x \le \left(\frac{2n+1}{2}\right)\pi,$$

   n an arbitrary positive integer? Give reasons for your answer.



33. **Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$.

34. **Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^{-x}$, and the line $x = 1$
   a. about the y-axis. **b.** about the line $x = 1$.

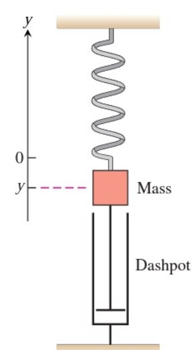
35. **Finding volume** Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes and the curve $y = \cos x$, $0 \le x \le \pi/2$, about
   a. the y-axis. **b.** the line $x = \pi/2$.

36. **Finding volume** Find the volume of the solid generated by revolving the region bounded by the x-axis and the curve $y = x \sin x$, $0 \le x \le \pi$, about
   a. the y-axis. **b.** the line $x = \pi$.
   (See Exercise 31 for a graph.)

37. **Average value** A retarding force, symbolized by the dashpot in the figure, slows the motion of the weighted spring so that the mass's position at time t is

$$y = 2e^{-t}\cos t, \qquad t \ge 0.$$

   Find the average value of y over the interval $0 \le t \le 2\pi$.



38. **Average value** In a mass-spring-dashpot system like the one in Exercise 37, the mass's position at time t is

$$y = 4e^{-t}(\sin t - \cos t), \qquad t \ge 0.$$

   Find the average value of y over the interval $0 \le t \le 2\pi$.

Reduction Formulas

In Exercises 39–42, use integration by parts to establish the *reduction formula*.

39. $\int x^n \cos x\,dx = x^n \sin x - n\int x^{n-1}\sin x\,dx$

40. $\int x^n \sin x\,dx = -x^n \cos x + n\int x^{n-1}\cos x\,dx$

41. $\int x^n e^{ax}\,dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx, \quad a \neq 0$

42. $\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$

Integrating Inverses of Functions

Integration by parts leads to a rule for integrating inverses that usually gives good results:

$$\int f^{-1}(x)\,dx = \int yf'(y)\,dy \qquad \begin{array}{l} y = f^{-1}(x), \quad x = f(y) \\ dx = f'(y)\,dy \end{array}$$

$$= yf(y) - \int f(y)\,dy \qquad \begin{array}{l} \text{Integration by parts with} \\ u = y,\, dv = f'(y)\,dy \end{array}$$

$$= xf^{-1}(x) - \int f(y)\,dy$$

The idea is to take the most complicated part of the integral, in this case $f^{-1}(x)$, and simplify it first. For the integral of $\ln x$, we get

$$\int \ln x\,dx = \int ye^y\,dy \qquad \begin{array}{l} y = \ln x, \quad x = e^y \\ dx = e^y\,dy \end{array}$$

$$= ye^y - e^y + C$$

$$= x\ln x - x + C.$$

For the integral of $\cos^{-1} x$ we get

$$\int \cos^{-1} x\,dx = x\cos^{-1} x - \int \cos y\,dy \qquad y = \cos^{-1} x$$

$$= x\cos^{-1} x - \sin y + C$$

$$= x\cos^{-1} x - \sin(\cos^{-1} x) + C.$$

Use the formula

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int f(y)\,dy \qquad y = f^{-1}(x) \qquad (4)$$

to evaluate the integrals in Exercises 43–46. Express your answers in terms of x.

43. $\int \sin^{-1} x\,dx$

44. $\int \tan^{-1} x\,dx$

45. $\int \sec^{-1} x\,dx$

46. $\int \log_2 x\,dx$

Another way to integrate $f^{-1}(x)$ (when f^{-1} is integrable, of course) is to use integration by parts with $u = f^{-1}(x)$ and $dv = dx$ to rewrite the integral of f^{-1} as

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int x\left(\frac{d}{dx}f^{-1}(x)\right)dx. \qquad (5)$$

Exercises 47 and 48 compare the results of using Equations (4) and (5).

47. Equations (4) and (5) give different formulas for the integral of $\cos^{-1} x$:

a. $\int \cos^{-1} x\,dx = x\cos^{-1} x - \sin(\cos^{-1} x) + C$ Eq. (4)

b. $\int \cos^{-1} x\,dx = x\cos^{-1} x - \sqrt{1 - x^2} + C$ Eq. (5)

Can both integrations be correct? Explain.

48. Equations (4) and (5) lead to different formulas for the integral of $\tan^{-1} x$:

a. $\int \tan^{-1} x\,dx = x\tan^{-1} x - \ln\sec(\tan^{-1} x) + C$ Eq. (4)

b. $\int \tan^{-1} x\,dx = x\tan^{-1} x - \ln\sqrt{1 + x^2} + C$ Eq. (5)

Can both integrations be correct? Explain.

Evaluate the integrals in Exercises 49 and 50 with **(a)** Eq. (4) and **(b)** Eq. (5). In each case, check your work by differentiating your answer with respect to x.

49. $\int \sinh^{-1} x\,dx$

50. $\int \tanh^{-1} x\,dx$

EXERCISES 8.3

Expanding Quotients into Partial Fractions

Expand the quotients in Exercises 1–8 by partial fractions.

1. $\dfrac{5x - 13}{(x - 3)(x - 2)}$

2. $\dfrac{5x - 7}{x^2 - 3x + 2}$

3. $\dfrac{x + 4}{(x + 1)^2}$

4. $\dfrac{2x + 2}{x^2 - 2x + 1}$

5. $\dfrac{z + 1}{z^2(z - 1)}$

6. $\dfrac{z}{z^3 - z^2 - 6z}$

7. $\dfrac{t^2 + 8}{t^2 - 5t + 6}$

8. $\dfrac{t^4 + 9}{t^4 + 9t^2}$

Nonrepeated Linear Factors

In Exercises 9–16, express the integrands as a sum of partial fractions and evaluate the integrals.

9. $\displaystyle\int \frac{dx}{1 - x^2}$

10. $\displaystyle\int \frac{dx}{x^2 + 2x}$

11. $\displaystyle\int \frac{x + 4}{x^2 + 5x - 6}\,dx$

12. $\displaystyle\int \frac{2x + 1}{x^2 - 7x + 12}\,dx$

13. $\displaystyle\int_4^8 \frac{y\,dy}{y^2 - 2y - 3}$

14. $\displaystyle\int_{1/2}^1 \frac{y + 4}{y^2 + y}\,dy$

15. $\displaystyle\int \frac{dt}{t^3 + t^2 - 2t}$

16. $\displaystyle\int \frac{x + 3}{2x^3 - 8x}\,dx$

Repeated Linear Factors

In Exercises 17–20, express the integrands as a sum of partial fractions and evaluate the integrals.

17. $\displaystyle\int_0^1 \frac{x^3\,dx}{x^2 + 2x + 1}$

18. $\displaystyle\int_{-1}^0 \frac{x^3\,dx}{x^2 - 2x + 1}$

19. $\displaystyle\int \frac{dx}{(x^2 - 1)^2}$

20. $\displaystyle\int \frac{x^2\,dx}{(x - 1)(x^2 + 2x + 1)}$

Irreducible Quadratic Factors

In Exercises 21–28, express the integrands as a sum of partial fractions and evaluate the integrals.

21. $\displaystyle\int_0^1 \frac{dx}{(x + 1)(x^2 + 1)}$

22. $\displaystyle\int_1^{\sqrt{3}} \frac{3t^2 + t + 4}{t^3 + t}\,dt$

23. $\displaystyle\int \frac{y^2 + 2y + 1}{(y^2 + 1)^2}\,dy$

24. $\displaystyle\int \frac{8x^2 + 8x + 2}{(4x^2 + 1)^2}\,dx$

25. $\displaystyle\int \frac{2s + 2}{(s^2 + 1)(s - 1)^3}\,ds$

26. $\displaystyle\int \frac{s^4 + 81}{s(s^2 + 9)^2}\,ds$

27. $\displaystyle\int \frac{2\theta^3 + 5\theta^2 + 8\theta + 4}{(\theta^2 + 2\theta + 2)^2}\,d\theta$

28. $\displaystyle\int \frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{(\theta^2 + 1)^3}\,d\theta$

Improper Fractions

In Exercises 29–34, perform long division on the integrand, write the proper fraction as a sum of partial fractions, and then evaluate the integral.

29. $\displaystyle\int \frac{2x^3 - 2x^2 + 1}{x^2 - x}\,dx$

30. $\displaystyle\int \frac{x^4}{x^2 - 1}\,dx$

31. $\displaystyle\int \frac{9x^3 - 3x + 1}{x^3 - x^2}\,dx$

32. $\displaystyle\int \frac{16x^3}{4x^2 - 4x + 1}\,dx$

33. $\displaystyle\int \frac{y^4 + y^2 - 1}{y^3 + y}\,dy$

34. $\displaystyle\int \frac{2y^4}{y^3 - y^2 + y - 1}\,dy$

Evaluating Integrals

Evaluate the integrals in Exercises 35–40.

35. $\displaystyle\int \frac{e^t\,dt}{e^{2t} + 3e^t + 2}$

36. $\displaystyle\int \frac{e^{4t} + 2e^{2t} - e^t}{e^{2t} + 1}\,dt$

37. $\displaystyle\int \frac{\cos y\,dy}{\sin^2 y + \sin y - 6}$

38. $\displaystyle\int \frac{\sin\theta\,d\theta}{\cos^2\theta + \cos\theta - 2}$

39. $\displaystyle\int \frac{(x - 2)^2\tan^{-1}(2x) - 12x^3 - 3x}{(4x^2 + 1)(x - 2)^2}\,dx$

40. $\displaystyle\int \frac{(x + 1)^2\tan^{-1}(3x) + 9x^3 + x}{(9x^2 + 1)(x + 1)^2}\,dx$

EXERCISES 8.4

Products of Powers of Sines and Cosines

Evaluate the integrals in Exercises 1–14.

1. $\displaystyle\int_0^{\pi/2} \sin^5 x\,dx$ **2.** $\displaystyle\int_0^{\pi} \sin^5 \frac{x}{2}\,dx$

3. $\displaystyle\int_{-\pi/2}^{\pi/2} \cos^3 x\,dx$ **4.** $\displaystyle\int_0^{\pi/6} 3\cos^5 3x\,dx$

5. $\displaystyle\int_0^{\pi/2} \sin^7 y\,dy$ **6.** $\displaystyle\int_0^{\pi/2} 7\cos^7 t\,dt$

7. $\displaystyle\int_0^{\pi} 8\sin^4 x\,dx$ **8.** $\displaystyle\int_0^{1} 8\cos^4 2\pi x\,dx$

9. $\displaystyle\int_{-\pi/4}^{\pi/4} 16\sin^2 x\cos^2 x\,dx$ **10.** $\displaystyle\int_0^{\pi} 8\sin^4 y\cos^2 y\,dy$

11. $\displaystyle\int_0^{\pi/2} 35\sin^4 x\cos^3 x\,dx$ **12.** $\displaystyle\int_0^{\pi} \sin 2x\cos^2 2x\,dx$

13. $\displaystyle\int_0^{\pi/4} 8\cos^3 2\theta\sin 2\theta\,d\theta$ **14.** $\displaystyle\int_0^{\pi/2} \sin^2 2\theta\cos^3 2\theta\,d\theta$

Integrals with Square Roots

Evaluate the integrals in Exercises 15–22.

15. $\displaystyle\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}}\,dx$ **16.** $\displaystyle\int_0^{\pi} \sqrt{1-\cos 2x}\,dx$

17. $\displaystyle\int_0^{\pi} \sqrt{1-\sin^2 t}\,dt$ **18.** $\displaystyle\int_0^{\pi} \sqrt{1-\cos^2 \theta}\,d\theta$

19. $\displaystyle\int_{-\pi/4}^{\pi/4} \sqrt{1+\tan^2 x}\,dx$ **20.** $\displaystyle\int_{-\pi/4}^{\pi/4} \sqrt{\sec^2 x-1}\,dx$

21. $\displaystyle\int_0^{\pi/2} \theta\sqrt{1-\cos 2\theta}\,d\theta$ **22.** $\displaystyle\int_{-\pi}^{\pi} (1-\cos^2 t)^{3/2}\,dt$

Powers of Tan x and Sec x

Evaluate the integrals in Exercises 23–32.

23. $\displaystyle\int_{-\pi/3}^{0} 2\sec^3 x\,dx$ **24.** $\displaystyle\int e^x\sec^3 e^x\,dx$

25. $\displaystyle\int_0^{\pi/4} \sec^4 \theta\,d\theta$ **26.** $\displaystyle\int_0^{\pi/12} 3\sec^4 3x\,dx$

27. $\displaystyle\int_{\pi/4}^{\pi/2} \csc^4 \theta\,d\theta$ **28.** $\displaystyle\int_{\pi/2}^{\pi} 3\csc^4 \frac{\theta}{2}\,d\theta$

29. $\displaystyle\int_0^{\pi/4} 4\tan^3 x\,dx$ **30.** $\displaystyle\int_{-\pi/4}^{\pi/4} 6\tan^4 x\,dx$

31. $\displaystyle\int_{\pi/6}^{\pi/3} \cot^3 x\,dx$ **32.** $\displaystyle\int_{\pi/4}^{\pi/2} 8\cot^4 t\,dt$

Products of Sines and Cosines

Evaluate the integrals in Exercises 33–38.

33. $\displaystyle\int_{-\pi}^{0} \sin 3x\cos 2x\,dx$ **34.** $\displaystyle\int_0^{\pi/2} \sin 2x\cos 3x\,dx$

35. $\displaystyle\int_{-\pi}^{\pi} \sin 3x\sin 3x\,dx$ **36.** $\displaystyle\int_0^{\pi/2} \sin x\cos x\,dx$

37. $\displaystyle\int_0^{\pi} \cos 3x\cos 4x\,dx$ **38.** $\displaystyle\int_{-\pi/2}^{\pi/2} \cos x\cos 7x\,dx$

Theory and Examples

39. Surface area Find the area of the surface generated by revolving the arc

$$x = t^{2/3},\quad y = t^2/2,\quad 0 \le t \le 2,$$

about the x-axis.

40. Arc length Find the length of the curve

$$y = \ln(\cos x),\quad 0 \le x \le \pi/3.$$

41. Arc length Find the length of the curve

$$y = \ln(\sec x),\quad 0 \le x \le \pi/4.$$

42. Center of gravity Find the center of gravity of the region bounded by the x-axis, the curve $y = \sec x$, and the lines $x = -\pi/4$, $x = \pi/4$.

43. Volume Find the volume generated by revolving one arch of the curve $y = \sin x$ about the x-axis.

44. Area Find the area between the x-axis and the curve $y = \sqrt{1+\cos 4x}$, $0 \le x \le \pi$.

45. Orthogonal functions Two functions f and g are said to be **orthogonal** on an interval $a \le x \le b$ if $\int_a^b f(x)g(x)\,dx = 0$.

a. Prove that $\sin mx$ and $\sin nx$ are orthogonal on any interval of length 2π provided m and n are integers such that $m^2 \ne n^2$.

b. Prove the same for $\cos mx$ and $\cos nx$.

c. Prove the same for $\sin mx$ and $\cos nx$ even if $m = n$.

46. Fourier series A finite Fourier series is given by the sum

$$f(x) = \sum_{n=1}^{N} a_n \sin nx = a_1\sin x + a_2\sin 2x + \cdots + a_N\sin Nx$$

Show that the mth coefficient a_m is given by the formula

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx.$$

EXERCISES 8.5

Basic Trigonometric Substitutions

Evaluate the integrals in Exercises 1–28.

1. $\int \frac{dy}{\sqrt{9+y^2}}$

2. $\int \frac{3\,dy}{\sqrt{1+9y^2}}$

3. $\int_{-2}^{2} \frac{dx}{4+x^2}$

4. $\int_{0}^{2} \frac{dx}{8+2x^2}$

5. $\int_{0}^{3/2} \frac{dx}{\sqrt{9-x^2}}$

6. $\int_{0}^{1/2\sqrt{2}} \frac{2\,dx}{\sqrt{1-4x^2}}$

7. $\int \sqrt{25-t^2}\,dt$

8. $\int \sqrt{1-9t^2}\,dt$

9. $\int \frac{dx}{\sqrt{4x^2-49}}, \quad x > \frac{7}{2}$

10. $\int \frac{5\,dx}{\sqrt{25x^2-9}}, \quad x > \frac{3}{5}$

11. $\int \frac{\sqrt{y^2-49}}{y}\,dy, \quad y > 7$

12. $\int \frac{\sqrt{y^2-25}}{y^3}\,dy, \quad y > 5$

13. $\int \frac{dx}{x^2\sqrt{x^2-1}}, \quad x > 1$

14. $\int \frac{2\,dx}{x^3\sqrt{x^2-1}}, \quad x > 1$

15. $\int \frac{x^3\,dx}{\sqrt{x^2+4}}$

16. $\int \frac{dx}{x^2\sqrt{x^2+1}}$

17. $\int \frac{8\,dw}{w^2\sqrt{4-w^2}}$

18. $\int \frac{\sqrt{9-w^2}}{w^2}\,dw$

19. $\int_{0}^{\sqrt{3}/2} \frac{4x^2\,dx}{(1-x^2)^{3/2}}$

20. $\int_{0}^{1} \frac{dx}{(4-x^2)^{3/2}}$

21. $\int \frac{dx}{(x^2-1)^{3/2}}, \quad x > 1$

22. $\int \frac{x^2\,dx}{(x^2-1)^{5/2}}, \quad x > 1$

23. $\int \frac{(1-x^2)^{3/2}}{x^6}\,dx$

24. $\int \frac{(1-x^2)^{1/2}}{x^4}\,dx$

25. $\int \frac{8\,dx}{(4x^2+1)^2}$

26. $\int \frac{6\,dt}{(9t^2+1)^2}$

27. $\int \frac{v^2\,dv}{(1-v^2)^{5/2}}$

28. $\int \frac{(1-r^2)^{5/2}}{r^8}\,dr$

In Exercises 29–36, use an appropriate substitution and then a trigonometric substitution to evaluate the integrals.

29. $\int_{0}^{\ln 4} \frac{e^t\,dt}{\sqrt{e^{2t}+9}}$

30. $\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t\,dt}{(1+e^{2t})^{3/2}}$

31. $\int_{1/12}^{1/4} \frac{2\,dt}{\sqrt{t}+4t\sqrt{t}}$

32. $\int_{1}^{e} \frac{dy}{y\sqrt{1+(\ln y)^2}}$

33. $\int \frac{dx}{x\sqrt{x^2-1}}$

34. $\int \frac{dx}{1+x^2}$

35. $\int \frac{x\,dx}{\sqrt{x^2-1}}$

36. $\int \frac{dx}{\sqrt{1-x^2}}$

Initial Value Problems

Solve the initial value problems in Exercises 37–40 for y as a function of x.

37. $x\frac{dy}{dx} = \sqrt{x^2-4}, \quad x \ge 2, \quad y(2) = 0$

38. $\sqrt{x^2-9}\,\frac{dy}{dx} = 1, \quad x > 3, \quad y(5) = \ln 3$

39. $(x^2+4)\frac{dy}{dx} = 3, \quad y(2) = 0$

40. $(x^2+1)^2\frac{dy}{dx} = \sqrt{x^2+1}, \quad y(0) = 1$

EXERCISES 8.8

Evaluating Improper Integrals

Evaluate the integrals in Exercises 1–34 without using tables.

1. $\displaystyle\int_0^{\infty} \frac{dx}{x^2+1}$

2. $\displaystyle\int_1^{\infty} \frac{dx}{x^{1.001}}$

3. $\displaystyle\int_0^{1} \frac{dx}{\sqrt{x}}$

4. $\displaystyle\int_0^{4} \frac{dx}{\sqrt{4-x}}$

5. $\displaystyle\int_{-1}^{1} \frac{dx}{x^{2/3}}$

6. $\displaystyle\int_{-8}^{1} \frac{dx}{x^{1/3}}$

7. $\displaystyle\int_0^{1} \frac{dx}{\sqrt{1-x^2}}$

8. $\displaystyle\int_0^{1} \frac{dr}{r^{0.999}}$

9. $\displaystyle\int_{-\infty}^{-2} \frac{2\,dx}{x^2-1}$

10. $\displaystyle\int_{-\infty}^{2} \frac{2\,dx}{x^2+4}$

11. $\displaystyle\int_2^{\infty} \frac{2}{v^2-v}\,dv$

12. $\displaystyle\int_2^{\infty} \frac{2\,dt}{t^2-1}$

13. $\displaystyle\int_{-\infty}^{\infty} \frac{2x\,dx}{(x^2+1)^2}$

14. $\displaystyle\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+4)^{3/2}}$

15. $\displaystyle\int_0^{1} \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\,d\theta$

16. $\displaystyle\int_0^{2} \frac{s+1}{\sqrt{4-s^2}}\,ds$

17. $\displaystyle\int_0^{\infty} \frac{dx}{(1+x)\sqrt{x}}$

18. $\displaystyle\int_1^{\infty} \frac{1}{x\sqrt{x^2-1}}\,dx$

19. $\displaystyle\int_0^{\infty} \frac{dv}{(1+v^2)(1+\tan^{-1} v)}$

20. $\displaystyle\int_0^{\infty} \frac{16\tan^{-1} x}{1+x^2}\,dx$

21. $\displaystyle\int_{-\infty}^{0} \theta e^{\theta}\,d\theta$

22. $\displaystyle\int_0^{\infty} 2e^{-\theta}\sin\theta\,d\theta$

23. $\displaystyle\int_{-\infty}^{0} e^{-|x|}\,dx$

24. $\displaystyle\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx$

25. $\displaystyle\int_0^{1} x\ln x\,dx$

26. $\displaystyle\int_0^{1} (-\ln x)\,dx$

27. $\displaystyle\int_0^{2} \frac{ds}{\sqrt{4-s^2}}$

28. $\displaystyle\int_0^{1} \frac{4r\,dr}{\sqrt{1-r^4}}$

29. $\displaystyle\int_1^{2} \frac{ds}{s\sqrt{s^2-1}}$

30. $\displaystyle\int_2^{4} \frac{dt}{t\sqrt{t^2-4}}$

31. $\displaystyle\int_{-1}^{4} \frac{dx}{\sqrt{|x|}}$

32. $\displaystyle\int_0^{2} \frac{dx}{\sqrt{|x-1|}}$

33. $\displaystyle\int_{-1}^{\infty} \frac{d\theta}{\theta^2+5\theta+6}$

34. $\displaystyle\int_0^{\infty} \frac{dx}{(x+1)(x^2+1)}$

Testing for Convergence

In Exercises 35–64, use integration, the Direct Comparison Test, or the Limit Comparison Test to test the integrals for convergence. If more than one method applies, use whatever method you prefer.

35. $\displaystyle\int_0^{\pi/2} \tan\theta\,d\theta$

36. $\displaystyle\int_0^{\pi/2} \cot\theta\,d\theta$

37. $\displaystyle\int_0^{\pi} \frac{\sin\theta\,d\theta}{\sqrt{\pi-\theta}}$

38. $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{\cos\theta\,d\theta}{(\pi-2\theta)^{1/3}}$

39. $\displaystyle\int_0^{\ln 2} x^{-2}e^{-1/x}\,dx$

40. $\displaystyle\int_0^{1} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

41. $\displaystyle\int_0^{\pi} \frac{dt}{\sqrt{t}+\sin t}$

42. $\displaystyle\int_0^{1} \frac{dt}{t-\sin t}$ (*Hint:* $t \ge \sin t$ for $t \ge 0$)

43. $\displaystyle\int_0^{2} \frac{dx}{1-x^2}$

44. $\displaystyle\int_0^{2} \frac{dx}{1-x}$

45. $\displaystyle\int_{-1}^{1} \ln|x|\,dx$

46. $\displaystyle\int_{-1}^{1} -x\ln|x|\,dx$

47. $\displaystyle\int_1^{\infty} \frac{dx}{x^3+1}$

48. $\displaystyle\int_4^{\infty} \frac{dx}{\sqrt{x}-1}$

49. $\displaystyle\int_2^{\infty} \frac{dv}{\sqrt{v-1}}$

50. $\displaystyle\int_0^{\infty} \frac{d\theta}{1+e^{\theta}}$

51. $\displaystyle\int_0^{\infty} \frac{dx}{\sqrt{x^6+1}}$

52. $\displaystyle\int_2^{\infty} \frac{dx}{\sqrt{x^2-1}}$

53. $\displaystyle\int_1^{\infty} \frac{\sqrt{x+1}}{x^2}\,dx$

54. $\displaystyle\int_2^{\infty} \frac{x\,dx}{\sqrt{x^4-1}}$

55. $\displaystyle\int_{\pi}^{\infty} \frac{2+\cos x}{x}\,dx$

56. $\displaystyle\int_{\pi}^{\infty} \frac{1+\sin x}{x^2}\,dx$

57. $\displaystyle\int_4^{\infty} \frac{2\,dt}{t^{3/2}-1}$

58. $\displaystyle\int_2^{\infty} \frac{1}{\ln x}\,dx$

59. $\displaystyle\int_1^{\infty} \frac{e^x}{x}\,dx$

60. $\displaystyle\int_{e^e}^{\infty} \ln(\ln x)\,dx$

61. $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{e^x-x}}\,dx$

62. $\displaystyle\int_1^{\infty} \frac{1}{e^x-2^x}\,dx$

63. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^4+1}}$

64. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{e^x+e^{-x}}$

Theory and Examples

65. Find the values of p for which each integral converges.

  a. $\displaystyle\int_1^{2} \frac{dx}{x(\ln x)^p}$ b. $\displaystyle\int_2^{\infty} \frac{dx}{x(\ln x)^p}$

66. **$\int_{-\infty}^{\infty} f(x)\,dx$ may not equal $\lim_{b\to\infty}\int_{-b}^{b} f(x)\,dx$** Show that

$$\int_0^{\infty} \frac{2x\,dx}{x^2+1}$$

diverges and hence that

$$\int_{-\infty}^{\infty} \frac{2x\,dx}{x^2+1}$$

diverges. Then show that

$$\lim_{b\to\infty}\int_{-b}^{b}\frac{2x\,dx}{x^2+1}=0.$$

Exercises 67–70 are about the infinite region in the first quadrant between the curve $y=e^{-x}$ and the x-axis.

67. Find the area of the region.

68. Find the centroid of the region.

69. Find the volume of the solid generated by revolving the region about the y-axis.

70. Find the volume of the solid generated by revolving the region about the x-axis.

71. Find the area of the region that lies between the curves $y=\sec x$ and $y=\tan x$ from $x=0$ to $x=\pi/2$.

72. The region in Exercise 71 is revolved about the x-axis to generate a solid.

a. Find the volume of the solid.

b. Show that the inner and outer surfaces of the solid have infinite area.

73. Estimating the value of a convergent improper integral whose domain is infinite

a. Show that

$$\int_3^\infty e^{-3x}\,dx=\frac{1}{3}e^{-9}<0.000042,$$

and hence that $\int_3^\infty e^{-x^2}\,dx<0.000042$. Explain why this means that $\int_0^\infty e^{-x^2}\,dx$ can be replaced by $\int_0^3 e^{-x^2}\,dx$ without introducing an error of magnitude greater than 0.000042.

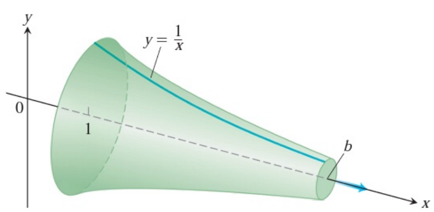
T **b.** Evaluate $\int_0^3 e^{-x^2}\,dx$ numerically.

74. The infinite paint can or Gabriel's horn As Example 3 shows, the integral $\int_1^\infty (dx/x)$ diverges. This means that the integral

$$\int_1^\infty 2\pi\frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx,$$

which measures the *surface area* of the solid of revolution traced out by revolving the curve $y=1/x$, $1\le x$, about the x-axis, diverges also. By comparing the two integrals, we see that, for every finite value $b>1$,

$$\int_1^b 2\pi\frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx>2\pi\int_1^b\frac{1}{x}\,dx.$$



However, the integral

$$\int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx$$

for the *volume* of the solid converges. **(a)** Calculate it. **(b)** This solid of revolution is sometimes described as a can that does not hold enough paint to cover its own interior. Think about that for a moment. It is common sense that a finite amount of paint cannot cover an infinite surface. But if we fill the horn with paint (a finite amount), then we *will* have covered an infinite surface. Explain the apparent contradiction.

75. Sine-integral function The integral

$$\text{Si}(x)=\int_0^x\frac{\sin t}{t}\,dt,$$

called the *sine-integral function,* has important applications in optics.

T **a.** Plot the integrand $(\sin t)/t$ for $t>0$. Is the Si function everywhere increasing or decreasing? Do you think $\text{Si}(x)=0$ for $x>0$? Check your answers by graphing the function $\text{Si}(x)$ for $0\le x\le 25$.

b. Explore the convergence of

$$\int_0^\infty\frac{\sin t}{t}\,dt.$$

If it converges, what is its value?

76. Error function The function

$$\text{erf}(x)=\int_0^x\frac{2e^{-t^2}}{\sqrt{\pi}}\,dt,$$

called the *error function*, has important applications in probability and statistics.

T **a.** Plot the error function for $0\le x\le 25$.

b. Explore the convergence of

$$\int_0^\infty\frac{2e^{-t^2}}{\sqrt{\pi}}\,dt.$$

If it converges, what appears to be its value? You will see how to confirm your estimate in Section 15.3, Exercise 37.

77. Normal probability distribution function The function

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

is called the *normal probability density function* with mean μ and standard deviation σ. The number μ tells where the distribution is centered, and σ measures the "scatter" around the mean.

From the theory of probability, it is known that

$$\int_{-\infty}^{\infty}f(x)\,dx=1.$$

In what follows, let $\mu=0$ and $\sigma=1$.

a. Draw the graph of f. Find the intervals on which f is increasing, the intervals on which f is decreasing, and any local extreme values and where they occur.

b. Evaluate

$$\int_{-n}^{n} f(x)\,dx$$

for $n = 1, 2, 3$.

c. Give a convincing argument that

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

(*Hint:* Show that $0 < f(x) < e^{-x/2}$ for $x > 1$, and for $b > 1$,

$$\int_{b}^{\infty} e^{-x/2}\,dx \rightarrow 0 \quad \text{as} \quad b \rightarrow \infty.)$$

✓ **78.** Here is an argument that ln 3 equals $\infty - \infty$. Where does the argument go wrong? Give reasons for your answer.

$$\begin{aligned}
\ln 3 &= \ln 1 + \ln 3 = \ln 1 - \ln \frac{1}{3} \\
&= \lim_{b\to\infty} \ln\left(\frac{b-2}{b}\right) - \ln\frac{1}{3} \\
&= \lim_{b\to\infty} \left[\ln \frac{x-2}{x}\right]_3^b \\
&= \lim_{b\to\infty} \Big[\ln(x-2) - \ln x\Big]_3^b \\
&= \lim_{b\to\infty} \int_3^b \left(\frac{1}{x-2} - \frac{1}{x}\right) dx \\
&= \int_3^\infty \left(\frac{1}{x-2} - \frac{1}{x}\right) dx \\
&= \int_3^\infty \frac{1}{x-2}\,dx - \int_3^\infty \frac{1}{x}\,dx \\
&= \lim_{b\to\infty} \Big[\ln(x-2)\Big]_3^b - \lim_{b\to\infty} \Big[\ln x\Big]_3^b \\
&= \infty - \infty.
\end{aligned}$$

79. Show that if $f(x)$ is integrable on every interval of real numbers and a and b are real numbers with $a < b$, then

a. $\int_{-\infty}^{a} f(x)\,dx$ and $\int_{a}^{\infty} f(x)\,dx$ both converge if and only if $\int_{-\infty}^{b} f(x)\,dx$ and $\int_{b}^{\infty} f(x)\,dx$ both converge.

b. $\int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx + \int_{b}^{\infty} f(x)\,dx$ when the integrals involved converge.

80. a. Show that if f is even and the necessary integrals exist, then

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\int_{0}^{\infty} f(x)\,dx.$$

b. Show that if f is odd and the necessary integrals exist, then

$$\int_{-\infty}^{\infty} f(x)\,dx = 0.$$

Use direct evaluation, the comparison tests, and the results in Exercise 80, as appropriate, to determine the convergence or divergence of the integrals in Exercises 81–88. If more than one method applies, use whatever method you prefer.

81. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^2+1}}$

82. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^6+1}}$

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83. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{e^x+e^{-x}}$

84. $\displaystyle\int_{-\infty}^{\infty} \frac{e^{-x}\,dx}{x^2+1}$

85. $\displaystyle\int_{-\infty}^{\infty} e^{-|x|}\,dx$

86. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x+1)^2}$

87. $\displaystyle\int_{-\infty}^{\infty} \frac{|\sin x| + |\cos x|}{|x|+1}\,dx$

(*Hint:* $|\sin\theta| + |\cos\theta| \geq \sin^2\theta + \cos^2\theta$.)

88. $\displaystyle\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+1)(x^2+2)}$

COMPUTER EXPLORATIONS

Exploring Integrals of $x^p \ln x$

In Exercises 89–92, use a CAS to explore the integrals for various values of p (include noninteger values). For what values of p does the integral converge? What is the value of the integral when it does converge? Plot the integrand for various values of p.

89. $\displaystyle\int_{0}^{e} x^p \ln x\,dx$

90. $\displaystyle\int_{e}^{\infty} x^p \ln x\,dx$

91. $\displaystyle\int_{0}^{\infty} x^p \ln x\,dx$

92. $\displaystyle\int_{-\infty}^{\infty} x^p \ln |x|\,dx$

f is even ⟺ $f(-x) = f(x)$ for all $x \in \mathbb{R}$

f is odd ⟺ $f(-x) = -f(x)$ for all $x \in \mathbb{R}$

Chapter 8 Practice Exercises

Integration Using Substitutions

Evaluate the integrals in Exercises 1–82. To transform each integral into a recognizable basic form, it may be necessary to use one or more of the techniques of algebraic substitution, completing the square, separating fractions, long division, or trigonometric substitution.

1. $\int x\sqrt{4x^2-9}\,dx$
2. $\int 6x\sqrt{3x^2+5}\,dx$
3. $\int x(2x+1)^{1/2}\,dx$
4. $\int x(1-x)^{-1/2}\,dx$
5. $\int \frac{x\,dx}{\sqrt{8x^2+1}}$
6. $\int \frac{x\,dx}{\sqrt{9-4x^2}}$
7. $\int \frac{y\,dy}{25+y^2}$
8. $\int \frac{y^3\,dy}{4+y^4}$
9. $\int \frac{t^3\,dt}{\sqrt{9-4t^4}}$
10. $\int \frac{2t\,dt}{t^4+1}$
11. $\int z^{2/3}(z^{5/3}+1)^{2/3}\,dz$
12. $\int z^{-1/5}(1+z^{4/5})^{-1/2}\,dz$
13. $\int \frac{\sin 2\theta\,d\theta}{(1-\cos 2\theta)^2}$
14. $\int \frac{\cos\theta\,d\theta}{(1+\sin\theta)^{1/2}}$
15. $\int \frac{\sin t}{3+4\cos t}\,dt$
16. $\int \frac{\cos 2t}{1+\sin 2t}\,dt$
17. $\int \sin 2x\, e^{\cos 2x}\,dx$
18. $\int \sec x\tan x\, e^{\sec x}\,dx$
19. $\int e^{\theta}\sin(e^{\theta})\cos^2(e^{\theta})\,d\theta$
20. $\int e^{\theta}\sec^2(e^{\theta})\,d\theta$
21. $\int 2^{x-1}\,dx$
22. $\int 5^{x\sqrt{2}}\,dx$
23. $\int \frac{dv}{v\ln v}$
24. $\int \frac{dv}{v(2+\ln v)}$
25. $\int \frac{dx}{(x^2+1)(2+\tan^{-1}x)}$
26. $\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$
27. $\int \frac{2\,dx}{\sqrt{1-4x^2}}$
28. $\int \frac{dx}{\sqrt{49-x^2}}$
29. $\int \frac{dt}{\sqrt{16-9t^2}}$
30. $\int \frac{dt}{\sqrt{9-4t^2}}$
31. $\int \frac{dt}{9+t^2}$
32. $\int \frac{dt}{1+25t^2}$
33. $\int \frac{4\,dx}{5x\sqrt{25x^2-16}}$
34. $\int \frac{6\,dx}{x\sqrt{4x^2-9}}$
35. $\int \frac{dx}{\sqrt{4x-x^2}}$
36. $\int \frac{dx}{\sqrt{4x-x^2-3}}$
37. $\int \frac{dy}{y^2-4y+8}$
38. $\int \frac{dt}{t^2+4t+5}$
39. $\int \frac{dx}{(x-1)\sqrt{x^2-2x}}$
40. $\int \frac{dv}{(v+1)\sqrt{v^2+2v}}$
41. $\int \sin^2 x\,dx$
42. $\int \cos^2 3x\,dx$
43. $\int \sin^3\frac{\theta}{2}\,d\theta$
44. $\int \sin^3\theta\cos^2\theta\,d\theta$
45. $\int \tan^3 2t\,dt$
46. $\int 6\sec^4 t\,dt$
47. $\int \frac{dx}{2\sin x\cos x}$
48. $\int \frac{2\,dx}{\cos^2 x-\sin^2 x}$
49. $\int_{\pi/4}^{\pi/2}\sqrt{\csc^2 y-1}\,dy$
50. $\int_{\pi/4}^{3\pi/4}\sqrt{\cot^2 t+1}\,dt$
51. $\int_0^{\pi}\sqrt{1-\cos^2 2x}\,dx$
52. $\int_0^{2\pi}\sqrt{1-\sin^2\frac{x}{2}}\,dx$
53. $\int_{-\pi/2}^{\pi/2}\sqrt{1-\cos 2t}\,dt$
54. $\int_{\pi}^{2\pi}\sqrt{1+\cos 2t}\,dt$
55. $\int \frac{x^2}{x^2+4}\,dx$
56. $\int \frac{x^3}{9+x^2}\,dx$
57. $\int \frac{4x^2+3}{2x-1}\,dx$
58. $\int \frac{2x}{x-4}\,dx$
59. $\int \frac{2y-1}{y^2+4}\,dy$
60. $\int \frac{y+4}{y^2+1}\,dy$
61. $\int \frac{t+2}{\sqrt{4-t^2}}\,dt$
62. $\int \frac{2t^2+\sqrt{1-t^2}}{t\sqrt{1-t^2}}\,dt$
63. $\int \frac{\tan x\,dx}{\tan x+\sec x}$
64. $\int \frac{\cot x}{\cot x+\csc x}\,dx$
65. $\int \sec(5-3x)\,dx$
66. $\int x\csc(x^2+3)\,dx$
67. $\int \cot\left(\frac{x}{4}\right)dx$
68. $\int \tan(2x-7)\,dx$
69. $\int x\sqrt{1-x}\,dx$
70. $\int 3x\sqrt{2x+1}\,dx$
71. $\int \sqrt{z^2+1}\,dz$
72. $\int (16+z^2)^{-3/2}\,dz$
73. $\int \frac{dy}{\sqrt{25+y^2}}$
74. $\int \frac{dy}{\sqrt{25+9y^2}}$
75. $\int \frac{dx}{x^2\sqrt{1-x^2}}$
76. $\int \frac{x^3\,dx}{\sqrt{1-x^2}}$
77. $\int \frac{x^2\,dx}{\sqrt{1-x^2}}$
78. $\int \sqrt{4-x^2}\,dx$
79. $\int \frac{dx}{\sqrt{x^2-9}}$
80. $\int \frac{12\,dx}{(x^2-1)^{3/2}}$
81. $\int \frac{\sqrt{w^2-1}}{w}\,dw$
82. $\int \frac{\sqrt{z^2-16}}{z}\,dz$

Integration by Parts

Evaluate the integrals in Exercises 83–90 using integration by parts.

83. $\int \ln(x+1)\,dx$
84. $\int x^2\ln x\,dx$
85. $\int \tan^{-1}3x\,dx$
86. $\int \cos^{-1}\left(\frac{x}{2}\right)dx$
87. $\int (x+1)^2e^x\,dx$
88. $\int x^2\sin(1-x)\,dx$
89. $\int e^x\cos 2x\,dx$
90. $\int e^{-2x}\sin 3x\,dx$

Partial Fractions

Evaluate the integrals in Exercises 91–110. It may be necessary to use a substitution first.

91. $\int \frac{x\,dx}{x^2-3x+2}$
92. $\int \frac{x\,dx}{x^2+4x+3}$
93. $\int \frac{dx}{x(x+1)^2}$
94. $\int \frac{x+1}{x^2(x-1)}\,dx$
95. $\int \frac{\sin\theta\,d\theta}{\cos^2\theta+\cos\theta-2}$
96. $\int \frac{\cos\theta\,d\theta}{\sin^2\theta+\sin\theta-6}$
97. $\int \frac{3x^2+4x+4}{x^3+x}\,dx$
98. $\int \frac{4x\,dx}{x^3+4x}$
99. $\int \frac{v+3}{2v^3-8v}\,dv$
100. $\int \frac{(3v-7)\,dv}{(v-1)(v-2)(v-3)}$
101. $\int \frac{dt}{t^4+4t^2+3}$
102. $\int \frac{t\,dt}{t^4-t^2-2}$
103. $\int \frac{x^3+x^2}{x^2+x-2}\,dx$
104. $\int \frac{x^3+1}{x^3-x}\,dx$
105. $\int \frac{x^3+4x^2}{x^2+4x+3}\,dx$
106. $\int \frac{2x^3+x^2-21x+24}{x^2+2x-8}\,dx$
107. $\int \frac{dx}{x(3\sqrt{x+1})}$
108. $\int \frac{dx}{x(1+\sqrt[3]{x})}$
109. $\int \frac{ds}{e^s-1}$
110. $\int \frac{ds}{\sqrt{e^s+1}}$

Trigonometric Substitutions

Evaluate the integrals in Exercises 111–114 **(a)** without using a trigonometric substitution, **(b)** using a trigonometric substitution.

111. $\int \frac{y\,dy}{\sqrt{16-y^2}}$
112. $\int \frac{x\,dx}{\sqrt{4+x^2}}$
113. $\int \frac{x\,dx}{4-x^2}$
114. $\int \frac{t\,dt}{\sqrt{4t^2-1}}$

Quadratic Terms

Evaluate the integrals in Exercises 115–118.

115. $\int \frac{x\,dx}{9-x^2}$
116. $\int \frac{dx}{x(9-x^2)}$
117. $\int \frac{dx}{9-x^2}$
118. $\int \frac{dx}{\sqrt{9-x^2}}$

Trigonometric Integrals

Evaluate the integrals in Exercises 119–126.

119. $\int \sin^3 x \cos^4 x \, dx$

120. $\int \cos^5 x \sin^5 x \, dx$

121. $\int \tan^4 x \sec^2 x \, dx$

122. $\int \tan^3 x \sec^3 x \, dx$

123. $\int \sin 5\theta \cos 6\theta \, d\theta$

124. $\int \cos 3\theta \cos 3\theta \, d\theta$

125. $\int \sqrt{1 + \cos (t/2)} \, dt$

126. $\int e^t \sqrt{\tan^2 e^t + 1} \, dt$

Numerical Integration

127. According to the error-bound formula for Simpson's Rule, how many subintervals should you use to be sure of estimating the value of

$$\ln 3 = \int_1^3 \frac{1}{x} \, dx$$

by Simpson's Rule with an error of no more than 10^{-4} in absolute value? (Remember that for Simpson's Rule, the number of subintervals has to be even.)

128. A brief calculation shows that if $0 \le x \le 1$, then the second derivative of $f(x) = \sqrt{1 + x^4}$ lies between 0 and 8. Based on this, about how many subdivisions would you need to estimate the integral of f from 0 to 1 with an error no greater than 10^{-3} in absolute value using the Trapezoidal Rule?

129. A direct calculation shows that

$$\int_0^\pi 2 \sin^2 x \, dx = \pi.$$

How close do you come to this value by using the Trapezoidal Rule with $n = 6$? Simpson's Rule with $n = 6$? Try them and find out.

130. You are planning to use Simpson's Rule to estimate the value of the integral

$$\int_1^2 f(x) \, dx$$

with an error magnitude less than 10^{-5}. You have determined that $|f^{(4)}(x)| \le 3$ throughout the interval of integration. How many subintervals should you use to assure the required accuracy? (Remember that for Simpson's Rule the number has to be even.)

131. Mean temperature Compute the average value of the temperature function

$$f(x) = 37 \sin \left(\frac{2\pi}{365} (x - 101) \right) + 25$$

for a 365-day year. This is one way to estimate the annual mean air temperature in Fairbanks, Alaska. The National Weather Service's official figure, a numerical average of the daily normal mean air temperatures for the year, is 25.7°F, which is slightly higher than the average value of $f(x)$.

132. Heat capacity of a gas Heat capacity C_v is the amount of heat required to raise the temperature of a given mass of gas with constant volume by 1°C, measured in units of cal/deg-mol (calories per degree gram molecular weight). The heat capacity of oxygen depends on its temperature T and satisfies the formula

$$C_v = 8.27 + 10^{-5} (26T - 1.87T^2).$$

Find the average value of C_v for $20° \le T \le 675°$C and the temperature at which it is attained.

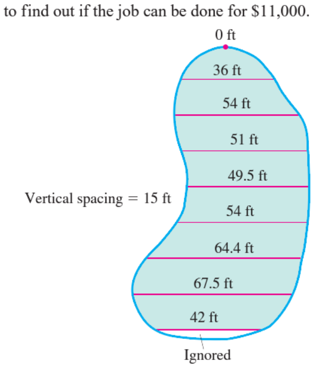
133. Fuel efficiency An automobile computer gives a digital readout of fuel consumption in gallons per hour. During a trip, a passenger recorded the fuel consumption every 5 min for a full hour of travel.

Time	Gal/h	Time	Gal/h
0	2.5	35	2.5
5	2.4	40	2.4
10	2.3	45	2.3
15	2.4	50	2.4
20	2.4	55	2.4
25	2.5	60	2.3
30	2.6		

a. Use the Trapezoidal Rule to approximate the total fuel consumption during the hour.

b. If the automobile covered 60 mi in the hour, what was its fuel efficiency (in miles per gallon) for that portion of the trip?

134. A new parking lot To meet the demand for parking, your town has allocated the area shown here. As the town engineer, you have been asked by the town council to find out if the lot can be built for \$11,000. The cost to clear the land will be \$0.10 a square foot, and the lot will cost \$2.00 a square foot to pave. Use Simpson's Rule to find out if the job can be done for \$11,000.



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Improper Integrals

Evaluate the improper integrals in Exercises 135–144.

135. $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

136. $\int_0^1 \ln x\, dx$

137. $\int_{-1}^1 \frac{dy}{y^{2/3}}$

138. $\int_{-2}^0 \frac{d\theta}{(\theta+1)^{3/5}}$

139. $\int_3^\infty \frac{2\, du}{u^2-2u}$

140. $\int_1^\infty \frac{3v-1}{4v^3-v^2}\, dv$

141. $\int_0^\infty x^2 e^{-x}\, dx$

142. $\int_{-\infty}^0 x e^{3x}\, dx$

143. $\int_{-\infty}^\infty \frac{dx}{4x^2+9}$

144. $\int_{-\infty}^\infty \frac{4dx}{x^2+16}$

✓

Convergence or Divergence

Which of the improper integrals in Exercises 145–150 converge and which diverge?

145. $\int_6^\infty \frac{d\theta}{\sqrt{\theta^2+1}}$

146. $\int_0^\infty e^{-u}\cos u\, du$

147. $\int_1^\infty \frac{\ln z}{z}\, dz$

148. $\int_1^\infty \frac{e^{-t}}{\sqrt{t}}\, dt$

149. $\int_{-\infty}^\infty \frac{2\, dx}{e^x+e^{-x}}$

150. $\int_{-\infty}^\infty \frac{dx}{x^2(1+e^x)}$

✓

Assorted Integrations

Evaluate the integrals in Exercises 151–218. The integrals are listed in random order.

151. $\int \frac{x\, dx}{1+\sqrt{x}}$

152. $\int \frac{x^3+2}{4-x^2}\, dx$

153. $\int \frac{dx}{x(x^2+1)^2}$

154. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx$

155. $\int \frac{dx}{\sqrt{-2x-x^2}}$

156. $\int \frac{(t-1)\, dt}{\sqrt{t^2-2t}}$

157. $\int \frac{du}{\sqrt{1+u^2}}$

158. $\int e^t \cos e^t\, dt$

159. $\int \frac{2-\cos x+\sin x}{\sin^2 x}\, dx$

160. $\int \frac{\sin^2\theta}{\cos^2\theta}\, d\theta$

161. $\int \frac{9\, dv}{81-v^4}$

162. $\int \frac{\cos x\, dx}{1+\sin^2 x}$

163. $\int \theta\cos(2\theta+1)\, d\theta$

164. $\int_2^\infty \frac{dx}{(x-1)^2}$

165. $\int \frac{x^3\, dx}{x^2-2x+1}$

166. $\int \frac{d\theta}{\sqrt{1+\sqrt{\theta}}}$

167. $\int \frac{2\sin\sqrt{x}\, dx}{\sqrt{x}\sec\sqrt{x}}$

168. $\int \frac{x^5\, dx}{x^4-16}$

169. $\int \frac{dy}{\sin y\cos y}$

170. $\int \frac{d\theta}{\theta^2-2\theta+4}$

171. $\int \frac{\tan x}{\cos^2 x}\, dx$

172. $\int \frac{dr}{(r+1)\sqrt{r^2+2r}}$

173. $\int \frac{(r+2)\, dr}{\sqrt{-r^2-4r}}$

174. $\int \frac{y\, dy}{4+y^4}$

175. $\int \frac{\sin 2\theta\, d\theta}{(1+\cos 2\theta)^2}$

176. $\int \frac{dx}{(x^2-1)^2}$

177. $\int_{\pi/4}^{\pi/2} \sqrt{1+\cos 4x}\, dx$

178. $\int (15)^{2x+1}\, dx$

179. $\int \frac{x\, dx}{\sqrt{2-x}}$

180. $\int \frac{\sqrt{1-v^2}}{v^2}\, dv$

181. $\int \frac{dy}{y^2-2y+2}$

182. $\int \ln\sqrt{x-1}\, dx$

183. $\int \theta^2\tan(\theta^3)\, d\theta$

184. $\int \frac{x\, dx}{\sqrt{8-2x^2-x^4}}$

185. $\int \frac{z+1}{z^2(z^2+4)}\, dz$

186. $\int x^3 e^{(x^2)}\, dx$

187. $\int \frac{t\, dt}{\sqrt{9-4t^2}}$

188. $\int_0^{\pi/10} \sqrt{1+\cos 5\theta}\, d\theta$

189. $\int \frac{\cot\theta\, d\theta}{1+\sin^2\theta}$

190. $\int \frac{\tan^{-1}x}{x^2}\, dx$

191. $\int \frac{\tan\sqrt{y}}{2\sqrt{y}}\, dy$

192. $\int \frac{e^t\, dt}{e^{2t}+3e^t+2}$

193. $\int \frac{\theta^2 d\theta}{4-\theta^2}$

194. $\int \frac{1-\cos 2x}{1+\cos 2x}\, dx$

195. $\int \frac{\cos(\sin^{-1}x)}{\sqrt{1-x^2}}\, dx$

196. $\int \frac{\cos x\, dx}{\sin^3 x-\sin x}$

197. $\int \sin\frac{x}{2}\cos\frac{x}{2}\, dx$

198. $\int \frac{x^2-x+2}{(x^2+2)^2}\, dx$

199. $\int \frac{e^t\, dt}{1+e^t}$

200. $\int \tan^3 t\, dt$

201. $\int_1^\infty \frac{\ln y}{y^3}\, dy$

202. $\int \frac{3+\sec^2 x+\sin x}{\tan x}\, dx$

203. $\int \frac{\cot v\, dv}{\ln\sin v}$

204. $\int \frac{dx}{(2x-1)\sqrt{x^2-x}}$

205. $\int e^{\ln\sqrt{x}}\, dx$

206. $\int e^\theta\sqrt{3+4e^\theta}\, d\theta$

207. $\int \frac{\sin 5t\, dt}{1+(\cos 5t)^2}$

208. $\int \frac{dv}{\sqrt{e^{2v}-1}}$

209. $\int (27)^{3\theta+1} d\theta$

210. $\int x^5\sin x\, dx$

211. $\int \frac{dr}{1+\sqrt{r}}$

212. $\int \frac{4x^3-20x}{x^4-10x^2+9}\, dx$

213. $\int \frac{8\, dy}{y^3(y+2)}$

214. $\int \frac{(t+1)\, dt}{(t^2+2t)^{2/3}}$

215. $\int \frac{8\, dm}{m\sqrt{49m^2-4}}$

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