3.8 กฎลูกโซ่ (Chain Rules)

ทบทวน! $y = f(x)$ และ $x = g(t)$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt} \left(= \frac{d(f(x))}{dx}\cdot\frac{d(g(t))}{dt} \right)$$

Chain Rule I:

ทฤษฎีบท: ถ้า $z = f(x,y)$ เป็นฟังก์ชันที่หาอนุพันธ์ได้และ $x = x(t)$ และ $y = y(t)$ เป็นฟังก์ชันที่หาอนุพันธ์ได้และต่อเนื่อง แล้ว

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

ตัวอย่าง: กำหนดให้ $z = x^2y$, $x = t^2$ และ $y = t^3$ จงหา $\frac{dz}{dt}$

วิธีทำ

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= \frac{\partial(x^2y)}{\partial x}\cdot\frac{d(t^2)}{dt} + \frac{\partial(x^2y)}{\partial y}\cdot\frac{d(t^3)}{dt}$$

$$= (2xy)(2t) + (x^2)(3t^2)$$

$$= 2(t^2t^3)(2t) + (t^4)(3t^2)$$
$$= 4t^6 + 3t^6 = 7t^6$$

ข้อตัวอย่าง: กำหนดให้ $z = \ln(2x^2+y)$; $x = \sqrt{t}$, $y = t^{2/3}$
และ จงหา $\dfrac{dz}{dt}$

วิธีทำ
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= \frac{\partial \ln(2x^2+y)}{\partial(2x^2+y)}\cdot\frac{\partial(2x^2+y)}{\partial x}\cdot\frac{d\sqrt{t}}{dt}$$

$$+ \frac{\partial \ln(2x^2+y)}{\partial(2x^2+y)}\cdot\frac{\partial(2x^2+y)}{\partial y}\cdot\frac{d\,t^{2/3}}{dt}$$

$$= \frac{1}{2x^2+y}(4x)\left(\frac{1}{2\sqrt{t}}\right)$$

$$+ \frac{1}{2x^2+y}(1)\left(\frac{2}{3}t^{-1/3}\right)$$

$$= \ldots$$

ข้อตัวอย่าง: กำหนดให้ $w = x\sin(yz^2)$, $x = \cos t$, $y = t^2$
และ $z = e^t$ จงหา $\dfrac{dw}{dt}$

วิธีทำ สูตร! $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\cdot\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\cdot\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\cdot\dfrac{dz}{dt}$

พิจารณา

$$\frac{\partial w}{\partial x} = \frac{\partial (x\sin(yz^2))}{\partial x} = \sin(yz^2)$$

$$\frac{dx}{dt} = \frac{d\cos t}{dt} = -\sin t$$

$$\frac{\partial w}{\partial y} = \frac{\partial (x\sin(yz^2))}{\partial y} = x\frac{\partial \sin(yz^2)}{\partial (yz^2)}\cdot\frac{\partial (yz^2)}{\partial y}$$

$$= x(\cos(yz^2))z^2$$

$$= xz^2\cos(yz^2)$$

$$\frac{dy}{dt} = \frac{dt^2}{dt} = 2t$$

$$\frac{\partial w}{\partial z} = \frac{\partial (x\sin(yz^2))}{\partial z} = x\frac{\partial (\sin(yz^2))}{\partial (yz^2)}\cdot\frac{\partial (yz^2)}{\partial z}$$

$$= x(\cos(yz^2))(2yz)$$

$$= 2xyz\cos(yz^2)$$

$$\frac{dz}{dt} = \frac{de^t}{dt} = e^t$$

เพราะฉะนั้น

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{\partial t} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

$$= (\sin(yz^2))(-\sin t) + (xz^2\cos(yz^2))(2t)$$

$$+ (2xyz\cos(yz^2))(e^t)$$

$= \ldots$

ฝาก! กำหนดให้ $w = x^3y^2z^4 ; x = t^2 , y = t+2$ และ $z = 2t^4$ จงหา $\dfrac{dw}{dt}$ $\square$

Chain Rule II:

ทฤษฎีบท: ถ้า $z = f(x,y)$ เป็นฟังก์ชันที่หาอนุพันธ์ได้ และ $x = g(s,t)$ และ $y = h(s,t)$ เป็นฟังก์ชันที่หาอนุพันธ์ได้และอนุพันธ์ย่อยเฉพาะที่เกี่ยวข้องหาค่าได้ แล้ว

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

ตัวอย่าง: กำหนดให้ $z = 8x^2y - 2x + 3y ; x = uv$ และ $y = u - v$ จงหา $\dfrac{\partial z}{\partial u}$ และ $\dfrac{\partial z}{\partial v}$

วิธีทำ

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= (16xy - 2)(v) + (8x^2 + 3)(1)$$

$$= 16uv(u-v)v - 2v + 8u^2v^2 + 3$$
$$= 16u^2v^2 - 16uv^3 - 2v + 8u^2v^2 + 3$$
$$= 24u^2v^2 - 16uv^3 - 2v + 3$$

$\frac{\partial z}{\partial v} = \ldots$ (ฝาก!) $\square$

ตัวอย่าง: กำหนดให้ $z = x^2 - y\tan x$; $x = \frac{u}{v}$ และ $y = u^2v^2$
จงหา $\frac{\partial z}{\partial u}$ และ $\frac{\partial z}{\partial v}$

วิธีทำ (ฝาก!)

ตัวอย่าง: กำหนดให้ $u = rs^2 \ln t$, $r = x^2$, $s = 4y+1$ และ $t = xy^3$
จงหา $\frac{\partial u}{\partial x}$ และ $\frac{\partial u}{\partial y}$

วิธีทำ $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

ตัวอย่าง: กำหนดให้ $w = 4x^2 + 4y^2 + z^2$
โดยที่ $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$
และ $z = \rho\cos\phi$
จงหา

$$\frac{\partial w}{\partial \rho}, \frac{\partial w}{\partial \phi}, \frac{\partial w}{\partial \theta}$$

วิธีทำ

$$\frac{\partial w}{\partial \rho} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \rho} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \rho} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial \rho}$$

$$\frac{\partial w}{\partial \phi} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \phi} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial \phi}$$

และ

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial \theta}$$

ตัวอย่าง: กำหนดให้ $w = s^2t + t^2u + u^2v + v^2s$

โดยที่ $s = x^2+y^2+z^2$, $t = xyz$, $u = \frac{xy}{2}$ และ $v = e^{xyz}$

จงหา

$$\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}$$

วิธีทำ

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial w}{\partial t}\cdot\frac{\partial t}{\partial x} + \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial x}$$

...

$\square$