ทบทวน! กำหนดให้ f เป็นฟังก์ชันหาอนุพันธ์ได้ และ

$$w = f(x-y,\ y-z,\ z-x)$$

จงแสดงว่า

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = 0$$

วิธีทำ

พิจารณา

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial (x-y)} \cdot \frac{\partial (x-y)}{\partial x} + \frac{\partial w}{\partial (y-z)} \cdot \frac{\partial (y-z)}{\partial x} + \frac{\partial w}{\partial (z-x)} \cdot \frac{\partial (z-x)}{\partial x}$$

$$= \frac{\partial w}{\partial (x-y)}(1) + \frac{\partial w}{\partial (y-z)}(0) + \frac{\partial w}{\partial (z-x)}(-1)$$

$$= \frac{\partial w}{\partial (x-y)} - \frac{\partial w}{\partial (z-x)}$$

และ

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial (x-y)} \cdot \frac{\partial (x-y)}{\partial y} + \frac{\partial w}{\partial (y-z)} \cdot \frac{\partial (y-z)}{\partial y} + \frac{\partial w}{\partial (z-x)} \cdot \frac{\partial (z-x)}{\partial y}$$

$$= \frac{\partial w}{\partial (x-y)}(-1) + \frac{\partial w}{\partial (y-z)}(1) + \frac{\partial w}{\partial (z-x)}(0)$$

$$= -\frac{\partial w}{\partial (x-y)} + \frac{\partial w}{\partial (y-z)}$$

และ

$$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial (x-y)} \cdot \frac{\partial (x-y)}{\partial z} + \frac{\partial w}{\partial (y-z)} \cdot \frac{\partial (y-z)}{\partial z} + \frac{\partial w}{\partial (z-x)} \cdot \frac{\partial (z-x)}{\partial t}$$

$$= 0 \quad - \frac{\partial w}{\partial (y-z)} + \frac{\partial w}{\partial (z-x)}$$

เพราะฉะนั้น

$$\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}+\frac{\partial w}{\partial z}=\cancel{\frac{\partial w}{\partial (x-y)}}-\cancel{\frac{\partial w}{\partial (z-x)}}-\cancel{\frac{\partial w}{\partial (x-y)}}+\cancel{\frac{\partial w}{\partial (y-z)}}-\cancel{\frac{\partial w}{\partial (y-z)}}+\cancel{\frac{\partial w}{\partial (z-x)}}$$

$$=0$$

$\square$

3.9 อนุพันธ์ย่อยของฟังก์ชันโดยปริยาย

พิจารณาฟังก์ชัน $x^3+y^2x-3=0$ จะหา $\frac{dy}{dx}$ ได้ดังนี้

$$\frac{d}{dx}\left(x^3+y^2x-3\right)=\frac{d}{dx}0$$

$$\Rightarrow \frac{dx^3}{dx}+\frac{d(y^2x)}{dx}-\frac{d3}{dx}=0$$

$$\Rightarrow 3x^2+y^2+x2y\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-3x^2-y^2}{2xy}$$

พิจารณาฟังก์ชัน $z=f(x,y)=0$ และ $y=y(x)$ จะได้

$$\frac{dz}{dx}=\frac{\partial f}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}=\frac{d0}{dx}$$

$$\Rightarrow \frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-\partial f/\partial x}{\partial f/\partial y}=-\frac{f_x}{f_y}$$

จาก $x^3+y^2x-3=0$

$\Rightarrow f_x = 3x^2+y^2$ และ $f_y = 2xy$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{f_x}{f_y} = \frac{-(3x^2+y^2)}{2xy} = \frac{-3x^2-y^2}{2xy}$$

$\square$

สูตร! จำได้ พิจารณาสมการ $z=f(x,y)=0$ โดยที่ $y=y(x)$

$$\boxed{\frac{dy}{dx} = \frac{-f_x}{f_y}}$$ Jum!

โดยที่ $f_y \neq 0$

ตัวอย่าง: กำหนดให้ $f(x,y) = xe^y + \sin xy + y = \ln 2$ จงหา $\frac{dy}{dx}$

วิธีทำ สูตร! $\frac{dy}{dx} = -\frac{f_x}{f_y}$

พิจารณา $f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(xe^y + \sin xy + y - \ln 2)$

$$= e^y + y\cos xy$$

และ

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(xe^y + \sin xy + y - \ln 2)$$

$$= xe^y + x\cos xy + 1$$

เพราะฉะนั้น $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}$

$$= \frac{-(e^y + y\cos xy)}{xe^y + x\cos xy + 1}$$

$$= \frac{-e^y - y\cos xy}{xe^y + x\cos xy + 1}$$

$\square$

<u>ตัวอย่าง</u>: กำหนดให้ $f(x,y) = \ln(x^2+y^2) - e^x \sin y$

จงหา $\dfrac{dy}{dx}$

<u>วิธีทำ</u> สูตร! $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}$

พิจารณา

$$f_x = \frac{\partial}{\partial x}\left(\ln(x^2+y^2) - e^x \sin y\right)$$

$$= \frac{1}{x^2+y^2}(2x) - e^x \sin y$$

$$= \frac{2x - e^x(x^2+y^2)\sin y}{x^2+y^2}$$

และ

$$f_y = \frac{\partial}{\partial y}\left(\ln(x^2+y^2) - e^x \sin y\right)$$

$$= \frac{1}{x^2+y^2}(2y) - e^x \cos y$$

$$= \frac{2y - e^{x}(x^2+y^2)\cos y}{x^2+y^2}$$

เพราะฉะนั้น

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{\dfrac{-2x + e^{x}(x^2+y^2)\sin y}{\cancel{(x^2+y^2)}}}{\dfrac{2y - e^{x}(x^2+y^2)\cos y}{\cancel{(x^2+y^2)}}}$$

$$= \frac{-2x + e^{x}(x^2+y^2)\sin y}{2y - e^{x}(x^2+y^2)\cos y}$$

$\square$

ตัวอย่าง: กำหนดให้ $f(x,y) = x^3 - 2y^2 + xy = 0$

จงหา $\dfrac{d^2y}{dx^2}$

วิธีทำ $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$

พิจารณา $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y} = \dfrac{-(3x^2+y)}{-4y+x} = \dfrac{3x^2+y}{4y-x}$

ดังนั้น

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{3x^2+y}{4y-x}\right)$$

$$= \frac{(4y-x)\left(6x + \frac{dy}{dx}\right) - (3x^2+y)\left(4\frac{dy}{dx} - 1\right)}{(4y-x)^2}$$

$= \left(\text{แทนค่า } \frac{dy}{dx}\right) \cdots$ $\square$

พิจารณา $w = f(x,y,z) = 0$ โดยที่ $z = z(x,y)$ จะได้

$$\frac{\partial w}{\partial x} = \frac{\partial 0}{\partial x}$$

$$\Rightarrow \quad \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial x} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} + 0 + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} = \frac{-f_x}{f_z}$$

และ

$$\frac{\partial z}{\partial y} = \frac{-f_y}{f_z}$$

พิจารณา สมการ $f(x,y,z) = 0$ โดยที่ $z = z(x,y)$
จะได้

$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z}$$ จำ! และ $$\frac{\partial z}{\partial y} = \frac{-f_y}{f_z}$$ จำ!

โดยที่ $f_z \neq 0$

ตัวอย่าง: กำหนดให้ $f(x,y,z) = \ln(1+z) + xy^2 + z = 1$

จงหา $\dfrac{\partial z}{\partial x}$ และ $\dfrac{\partial z}{\partial y}$

วิธีทำ สูตร! $\dfrac{\partial z}{\partial x} = -\dfrac{f_x}{f_z}$

พิจารณา $$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\ln(1+z) + xy^2 + z - 1\right)$$
$$= 0 + y^2 + 0 + 0$$

และ $$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(\ln(1+z) + xy^2 + z - 1\right)$$
$$= \frac{1}{1+z} + 0 + 1 + 0$$

ทำให้ $$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z} = \frac{-y^2}{\frac{1}{1+z}+1} = \cdots$$

สูตร! $$\frac{\partial z}{\partial y} = -\frac{f_y}{f_z} = \frac{-(2xy)}{\frac{1}{1+z}+1}$$

$\square$

ตัวอย่าง: กำหนดให้ $f(x,y,z) = e^{xy}\cos yz + e^{yz}\sin xz = 2$

จงหา $\dfrac{\partial z}{\partial x}$ และ $\dfrac{\partial z}{\partial y}$

@ กำหนดให้ $f(x,y,z) = \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1$

จงหา $\left.\dfrac{\partial z}{\partial x}\right|_{(2,3,6)}$ และ $\left.\dfrac{\partial z}{\partial y}\right|_{(2,3,6)}$

3 กำหนดให้ $f(x,y,z) = \sin(x+y)+\sin(y+z)+\sin(x+z)=0$

จงหา $\left.\dfrac{\partial z}{\partial x}\right|_{(\pi,\pi,\pi)}$ และ $\left.\dfrac{\partial z}{\partial y}\right|_{(\pi,\pi,\pi)}$